FINE STRUCTURE THEORY OF THE CONSTRUCTIBLE UNIVERSE IN α- AND β-RECURSION THEORY

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Priority arguments and arguments from the fine structure theory of L - Gödel's universe of constructible sets - meet together in α- and β-recursion theory. In this paper we stress the contributions of set theory and try to give an idea how the fine structure of L is used in α- and β-recursion theory. The reader should be familiar with L but nearly no prerequisites from recursion theory are required since all the notions we need are defined in terms of set theory and a short introduction to priority arguments is given in the beginning. Since we concentrate on a demonstration of typical methods rather than on a description of the variety of results the reader should consult for the latter the survey papers by S.D.Friedman, M.Lerman, R.A.Shore and S.G. Simpson which will appear in the Proceedings of the Second Symposium on Generalized Recursion Theory in Oslo 1977.

We restrict our attention to the existence of incomparable degrees and consider corresponding constructions in ordinary recursion theory (ORT), α-recursion theory (α is always an admissible ordinal in the following) and β-recursion theory (β is always a limit ordinal in the following which may be inadmissible). In course of this one can observe how the priority arguments from ORT tend to collapse and arguments from the fine structure theory of L take over. The use of set theoretical methods makes it then possible to prove for inadmissible β results without counterpart in ORT. As a new result we will construct in this paper incomparable β-recursive degrees for many inadmissible β.

For limit ordinals β a set $A \subseteq L_\beta$ is called <u>β-recursively enumerable</u> (β-r.e.) if A is definable over L_β by some Σ_1 - formula which may contain elements of L_β as parameters (we write then: A is Σ_1 L_β). If A and L_β - A are β-r.e. then A is <u>β-recursive</u>. Sets that are elements of L_β are called <u>β-finite</u> and we reserve the letters K , H for β-finite sets.

One further defines for $A,B \subseteq L_\beta$ that $A \leq_\beta B$ (" <u>A is β-reducible to B</u> ") if there is a β-r.e. set W_e (we fix an universal Σ_1 L_β predicate U^β and write W_e for $U^\beta(e,\cdot)$) such that the following two equivalences hold :

$$K \subseteq A \leftrightarrow \exists H_1 H_2 (\langle 0,K,H_1,H_2\rangle \in W_e \wedge H_1 \subseteq B \wedge H_2 \subseteq L_\beta - B)$$

$$K \subseteq L_\beta - A \leftrightarrow \exists H_1 H_2 (\langle 1,K,H_1,H_2\rangle \in W_e \wedge H_1 \subseteq B \wedge H_2 \subseteq L_\beta - B) .$$

One often communicates the index e by writing $A \leq_\beta^e B$.

The equivalence relation $=_\beta$ is defined by

$A =_\beta B :\Leftrightarrow A \leq_\beta B \wedge B \leq_\beta A$ and the equivalence classes with respect to $=_\beta$ are called β-degrees.

For the special case $\beta = \omega$ these are the basic notions of ORT. Observe that for any limit ordinal β there is really an enumeration procedure for β-r.e. sets : If A is defined over L_β by the Σ_1 -formula Ψ we generate mechanically $L_1, L_2, \ldots, L_\gamma, \ldots$ ($\gamma < \beta$) and enumerate at stage γ of this process those x into A for which $L_\gamma \models \Psi(x)$ becomes true (in the following we will often write $L_\gamma \models [x \in A]$ instead of $L_\gamma \models \Psi(x)$ for some fixed Σ_1 definition Ψ). $A \leq_\omega B$ means of course that A is Turing reducible to B , i.e. A can be substituted by B as an oracle for Turing machines. The given generalization $\leq_\beta$ is intuitively justified if one insists that every single computation in β-recursion theory is a β-finite object.

The way from ω to β took some time and was done in several steps. Kreisel and Sacks considered the case $\beta = \omega_1^{CK}$ ("meta recursion theory") where one has that the ω_1^{CK}-r.e. subsets of ω are just the Π_1^1 sets. The notion of an admissible ordinal was introduced by Kripke and Platek in order to get a class of ordinals where the associated recursion theory has many common features, e.g. if K is α-finite and f is an α-recursive function such that $K \subseteq \operatorname{dom} f$ then $f[K]$ is again α-finite. An enormous amount of papers has been written on α-recursion theory, among others by Lerman, Sacks, Shore and Simpson. Much of its attraction is due to the fact that the basic notions and most of the easy results of ORT can immediately be transferred to α-recursion theory. Thus α-

recursion theory is the canonical place to study the deeper parts of ORT in a generalized context and the results show that interesting distinctions and phenom na occur. β-recursion theory was started by S.Friedman and Sacks [2] , [3] . Many simple facts from ORT are not true in inadmissible recursion theory (e.g. the replacement scheme which we mentioned for α-recursion theory). But many of these facts may be more or less accidental so that it remains to explore the "hard core" of recursion theory without admissibility. In addition β-recursion theory helps to understand some parts of α-recursion theory (see e.g. [7]).

Post asked in 1944 whether there are r.e. sets which are neither in the degree 0 (the degree of the empty set) nor in the degree 0' (the degree of the universal Σ_1 L_ω predicate U^ω).

Post's problem was solved in 1956 by Friedberg and Muchnik who invented the priority method and constructed two r.e. sets A and B which are incomparable with respect to $\leq_\omega$. The sets A and B are enumerated during an effective process in ω steps and we write A_σ for the set of elements which have been put into A before step σ (analogous for B). We fix an enumeration of the universal Σ_1 L_ω predicate so that the notation $W_{e,\sigma}$ makes sense. During the construction one tries to satisfy for every $e \in \omega$ the requirements $R_e{}^A :\equiv \neg A \leq_\omega^e B$ and $R_e{}^B :\equiv \neg B \leq_\omega^e A$. A requirement $R_e{}^A$ is satisfied by establishing a counterexample to the relation $A \leq_\omega^e B$. We try to make x a counterexample to the relation $A \leq_\omega^e B$ at step σ of the construction if

$$\exists\ H_1\ H_2 \in L_\sigma\ (\langle 1, x ,H_1,H_2 \rangle \in W_{e,\sigma} \wedge H_1 \subseteq B_\sigma \wedge H_2 \subseteq L_\omega - B_\sigma)$$

in which case we put x into A at step σ. We promise then at step σ to keep all elements of H_2 out of B and if we don't injure this promise at a later step by putting an element of H_2 into B we have made x a real counterexample to $A \leq_\omega^e B$. A conflict arises because at some step $\sigma' > \sigma$ we may want to satisfy some requirement $R_{e'}{}^B$ in an analogous way (with A and B interchanged) by putting a suitable x' into B and it might happen that $x' \in H_2$. The conflict is solved according to the "priorities" 2e respectively 2e'+1 of these requirements $R_e{}^A$ and $R_{e'}{}^B$: If $2e < 2e'+1$

we don't put x' into B at step σ' so that a new attempt with some $x'' \geq \sup H_2$ has to be made at some later stage in order to satisfy $R_{e'}{}^B$; if $2e \geq 2e'+1$ we put x' into B at step σ' so that a new attempt has to be made in order to satisfy $R_e{}^A$ at a later step.

Of course we need not make a new attempt for any requirement as long as the promise associated with an earlier attempt for this requirement is not yet injured. So the construction is designed in such a way that the following holds (we write R_{2e} for $R_e{}^A$ and $R_{2e'+1}$ for $R_{e'}{}^B$) :

Priority Lemma : If σ is such that after step σ no attempt is made in order to satisfy a requirement R_e with $e < \tilde{e}$ then there is a step $\tilde{\sigma} \geq \sigma$ such that after step $\tilde{\sigma}$ no attempt is made in order to satisfy $R_{\tilde{e}}$.

With the help of the Priority Lemma one can prove by induction on e the crucial fact that for every $e \in \omega$ there is a $\sigma_e < \omega$ such that no attempt for some $R_{e'}$ with $e' < e$ is made after step σ_e .

If there exists an attempt for $R_e{}^A$ at step σ_{2e} where the associated promise is not yet injured or if a new attempt for $R_e{}^A$ is made at a step after σ_{2e} then the promise associated with this attempt for $R_e{}^A$ will never be injured according to the definition of σ_{2e} . Thus a counterexample to the relation $A \leq_{\omega}^{e} B$ is successfully established. A moment's thought shows that in case that $A \leq_{\omega}^{e} B$ holds we would always be able to make a new attempt for $R_e{}^A$ after step σ_{2e} if necessary (consider $x \in \omega - A$; it is easy to make sure that $\omega - A$ is unbounded). Thus the desired incomparability of A and B is proved.

There is no problem to do a similar construction for any admissible α such that an analogous version of the Priority Lemma holds (the construction is of course done in α steps and we have to consider requirements R_e for every $e \in \alpha$). But if we consider then as in ORT for every $e \in \alpha$ the least step σ_e such that after this step no attempt for any $R_{e'}$ with $e' < e$ is made we have problems to show as in ORT by induction on e that $\sigma_e < \alpha$ exists. The problem occurs at limit points $\lambda < \alpha$ of the priority list where we have to show first that the set $\{\sigma_e \mid e < \lambda\}$ is bounded below α before we can apply the Priority Lemma and get $\sigma_\lambda < \alpha$. There exists of course a bound for $\{\sigma_e \mid e < \lambda\}$ if α is Σ_2 -admissible since the function $e \mapsto \sigma_e$ is Σ_2 L_α . Σ_1 -admissibility is not enough since this function can't be Σ_1 L_α .

Nevertheless Sacks and Simpson [9] constructed for every admissible α α-r.e. sets A , B of incomparable degree. Their construction is designed in consideration of the fine structure of L_α in such a way that the growth of the Σ_2 L_α function can be controlled without the axiom of Σ_2 - admissibility. We will give here a short sketch of their construction since notions and arguments from the fine structure theory of L are essentially involved.

Define for any limit ordinal β the Σ_n -projectum $\sigma n p \beta$ of β (one usually writes β^* for $\sigma 1 p \beta$ in recursion theory) to be the least $\delta \leq \beta$ such that some Σ_n L_β function maps β 1-1 into δ .

Since there exist parameter free Δ_1 L_β well-orderings $<_\beta$ of L_β for every β we can get a parameter free Σ_1 L_β definable Σ_1 skolem function h_β for every β . h_β is a partial function $L_\beta \to L_\beta$ and if $X \subseteq L_\beta$ is closed under pairing and $\omega \subseteq X$ then we have $h[X] \prec_{\Sigma_1} L_\beta$; in fact $h[X]$ is the least Σ_1 -elementary substructure of L_β which contains X .

We will often use Jensen's Condensation Lemma [1] which says that for every Σ_1 substructure $S \prec_{\Sigma_1} L_\beta$ the transitive collapse of S has the form L_γ for some $\gamma \leq \beta$.

One says that $\varkappa \in \beta$ is a (regular) β-cardinal if $L_\beta \models [\varkappa$ is a (regular) cardinal$]$ and $\varkappa$ is the β-cardinality of some $x \in L_\beta$ if $L_\beta \models [\varkappa = \mathrm{card}(x)]$.

For admissible α regular α-cardinals behave with respect to α-recursive functions like "real" regular cardinals. An important example is the following Lemma due to Sacks and Simpson (Lemma 2.3. in [9]). Consider for a proof α-recursive enumerations of the involved α-r.e. sets and use the admissibility of α.

Lemma : Assume that α is admissible, $\varkappa$ is an infinite regular α-cardinal, $\delta \lesssim \varkappa$ and the sequence $\langle K_e \mid e < \delta \rangle$ of α-finite sets K_e of α-cardinality less than $\varkappa$ is simultaneously α-r.e. (i.e. the set $\{\langle e,x\rangle \mid e < \delta \wedge x \in K_e\}$ is α-r.e.). Then $\bigcup\{K_e \mid e < \delta\}$ is α-finite and of α-cardinality less than $\varkappa$.

This Lemma is used in the proof of Sacks and Simpson in order to show the existence of the bounds σ_e for every priority $e \in \alpha$. K_e is defined to be the set of steps in the construction where an attempt is made for requirement R_e (this is a refinement of the argument in ORT where it was sufficient to consider just $\sup K_e$). One shows then by induction on e that for every $e < \varkappa$ the set K_e is α-finite and of α-cardinality less than $\varkappa$ if $\varkappa$ is a regular α-cardinal. At successor stages of this inductive argument it is used that the only situation where a new attempt for some requirement R_e has to be made is the following : After an earlier attempt for R_e an attempt for some requirement $R_{e'}$ with $e' < e$ was made which injured the promise associated with the earlier attempt for R_e. The inductive argument doesn't brake down at limit stages because the preceding Lemma can be used at this point. Thus the construction from ORT works as well if <u>α is the limit of α-cardinals</u>. More sophisticated constructions have to be used for the other α.

In case that <u>$\alpha^* < \alpha$</u> there is an escape which became standard in α-recursion theory : we make a shorter priority list by fixing a 1-1 α-recursive map f from α into α^*. Then requirement R_e gets the priority $f(e)$ and we can essentially argue as before since α^* is an α-cardinal (define here K_e for $e < \alpha^*$ to be the set of steps where an attempt is made for requirement $R_{f^{-1}(e)}$ if $e \in \mathrm{Rg}\, f$, define $K_e := \emptyset$ otherwise). A technical problem occurs in this approach since $\mathrm{Rg}\, f$ can't be α-finite (otherwise f^{-1} would be a counterexample to the admissibility of α). But it can be shown that

$Rg\, f \cap \gamma \in L_\alpha$ for every $\gamma < \alpha^*$ by taking a suitable Σ_1 skolem hull and considering the transitive collapse of it. This is sufficient since one has then for every $\gamma < \alpha^*$ that $f \cap (\alpha \times \gamma)$ is known during the construction from some stage σ_γ on.

For the final and least trivial case where $\alpha^* = \alpha$ and there is a greatest α-cardinal $\kappa < \alpha$ one observes that then the set of δ such that $L_\delta \prec_{\Sigma_1} L_\alpha$ is unbounded in α (Take for a proof the Σ_1 skolem hull S of some L_γ in L_α where $\gamma > \kappa$. S is transitive since κ is the greatest α-cardinal, thus $S = L_\delta$ for some $\delta \leq \alpha$ by the Collapsing Lemma. We have in fact $\delta < \alpha$ since $\delta = \alpha$ would imply that $\alpha^* \leq \gamma$.). Consider now some $L_\gamma \prec_{\Sigma_1} L_\alpha$ such that $\kappa < \gamma < \alpha$. Then γ is admissible, $\gamma^* = \kappa$ and κ is the greatest γ-cardinal which is exactly the (less difficult) situation which was considered (for α instead of γ) in the preceding case. One iterates then the previous trick for every such γ and uses a Σ_1 L_γ function f_γ which maps γ 1-1 into κ to give priorities for requirements R_e with $e \in \gamma$. Of course one doesn't know at the beginning of the construction in advance which the Σ_1 substructures L_γ of L_α are but at any step $\sigma > \gamma$ of the construction (observe that at step σ one may essentially use just the information which is available in L_σ in order to keep the constructed sets A , B α-r.e.) one knows for which $\delta \leq \gamma$ $L_\delta \prec_{\Sigma_1} L_\alpha$ holds and one has the (somehow canonically chosen) Σ_1 L_γ projection f_γ at hand in order to attach priorities to the R_e with $e < \gamma$. One performs then the argument from the previous case for γ' instead of α where $L_{\gamma'}$ is the next Σ_1 substructure of L_α after L_γ. The argument shows here that K_e^γ , the set of steps between γ and γ' where an attempt for $R_{f_\gamma^{-1}(e)}$ is made, is for every $e < \kappa$ γ'-finite and of γ'-cardinality less than κ ; thus K_e^γ is in particular bounded below γ' by some σ_e^γ . But since $L_{\gamma'} \prec_{\Sigma_1} L_\alpha$ we have that no further attempt will be made for $R_{f_\gamma^{-1}(e)}$ at any step after γ' so that σ_e^γ is in fact already the final bound below α for the attempts which are made in order to satisfy $R_{f_\gamma^{-1}(e)}$.

Thus the existence of incomparable α-r.e. degrees is shown for every admissible α .

In order to get further information about the structure of α-r.e. degrees and α-r.e. sets more complicated constructions are necessary which usually require the use of the Σ_2 projectum of α ($\sigma 2p\,\alpha$). This occurs if the argument requires that every Σ_2 L_α subset of an initial segment of the priority list is α-finite. Thus the priority list must be no longer than

$\rho_{2,\alpha} := \mu\,\delta \leq \alpha$ (there exists some Σ_2 L_α set $S \subseteq \delta$ such that S is not α-finite) .

For admissible α one can see immediately that Σ_2 L_α definable Σ_2 skolem functions exist and by using this fact it can be shown that $\rho_{2,\alpha} = \sigma 2p\,\alpha$ (consider the transitive collapse of a suitable Σ_2 skolem hull). It is just this equality which makes then the priority construction possible since we can use a Σ_2 projection of α into $\rho_{2,\alpha}$ analogously as before in order to attach priorities whereas usually we couldn't use a Σ_3 L_α function for this purpose because α-recursive approximations to such a function don't converge good enough (we have to use α-recursive approximations during the construction because the construction itself must of course remain α-recursive).

The sketched proof for the equality $\rho_{2,\alpha} = \sigma 2p\,\alpha$ used the admissibility of α (only for admissible α Σ_2 L_α definable Σ_2 skolem functions are easily available). But it was shown by Jensen in the Uniformization Theorem [1] that one can prove in fact for every limit ordinal β that $\rho_{n,\beta} = \sigma np\,\beta$ for every $n \geq 1$ by using a more sophisticated argument (mastercodes). It is this situation which one hopes to find again in recursion theory : That admissibility is a useful but in principal unnecessary assumption for some basic constructions of recursion theory.

There is another point where the example of Jensen's Uniformization Theorem is instructive : Although the result is uniform for all β there is one step in the proof where one has to argue by cases depending on whether some structure is strongly inadmissible (see definition below) or not, namely Lemma 12 on p.93 in [1] . For the strongly inadmissible case one uses an argument which wouldn't work for the admissible case since the inadmissibility is used in a "positive" way: "There exists a certain β-recursive cofinal function.." (see Remark 2) at the end of this paper). Therefore it is useful to be open minded in the step from α- to β-recursion theory and to look for typical new effects besides trying to recover the familiar ones.

One of the most surprising new features of <u>inadmissible recursion theory</u> is the appearance of β-recursive (i.e. Δ_1 L_β) sets which are not in the degree 0 , the β-degree of the empty set (S.Friedman [2] [4]) : If q is a β-recursive function which maps some $\delta < \beta$ cofinally into β and U^β is an universal Σ_1 L_β predicate (thus $U^\beta \in 0'$) then the β-degree r of the β-recursive set

$$\{ \langle \gamma, x \rangle \mid L_{q(\gamma)} \models [x \in U^\beta] \}$$

lies strictly between 0 and $0'$ and r is an upper bound for all β-recursive degrees. Observe that we are following the usual convention and say that a degree has a certain property if the degree contains a set with this property.

The existence of a β-recursive degree $r > 0$ is not a contradiction since if a set is β-recursive this tells us something about the definability of the set whereas the β-degree of a set gives us information about the behaviour of this set as an oracle. $A \leq_\beta B$ means that A can always be substituted by B as an oracle and β-

degrees are just equivalence classes of oracles. The relation " $A =_\beta B$ " occurs as well outside of recursion theory : If the sets $A, B \subseteq L_\beta$ are regular over L_β (i.e. $\forall \gamma < \beta (A \cap L_\gamma \in L_\beta)$) then we have that $A =_\beta B$ is equivalent to

for every $n > 0$ and every $D \subseteq L_\beta$ D is $\Sigma_n \langle L_\beta, A \rangle$ iff D is $\Sigma_n \langle L_\beta, B \rangle$.

For any limit β one writes <u>$\sigma 1 \mathrm{cf}\, \beta$</u> for the least $\delta \leq \beta$ such that a β-recursive cofinal function from δ into β exists. We have that $\sigma 1 \mathrm{cf}\, \beta < \beta$ iff β is inadmissible. Inadmissible recursion theory tends to split into two cases :

$\beta^* \leq \sigma 1 \mathrm{cf}\, \beta < \beta$ (β is <u>"weakly inadmissible"</u>) and

$\beta^* > \sigma 1 \mathrm{cf}\, \beta$ (β is <u>"strongly inadmissible"</u>)

(observe that one has always $\beta^* < \beta$ for inadmissible β [4]).

S.Friedman [2] , [4] , [5] showed that incomparable β-r.e. degrees exist if β is weakly inadmissible or if β is strongly inadmissible and β^* is a <u>regular</u> β-cardinal. We will show in the following that for these β there exist in fact <u>β-recursive</u> degrees which are incomparable. Thus it seems to be a typical phenomenon of inadmissible recursion theory that β-recursive degrees behave like r.e. degrees in the admissible case.

For <u>weakly inadmissible β</u> the existence of incomparable β-recursive degrees was shown in [6] by using a collapsing argument. If β is weakly inadmissible then there is an admissible structure

$\mathfrak{A} := \langle L_{\sigma 1 \mathrm{cf}\, \beta}, T \rangle$, the <u>"admissible collapse of L_β"</u>

such that one has for every set $A \subseteq L_{\sigma 1 \mathrm{cf}\, \beta}$ that A is $\Sigma_1\ L_\beta$ iff A is $\Sigma_1\ \mathfrak{A}$. So far this approach is in the line of the collapse which was introduced by Jensen as a tool in the fine structure theory of L : He collapsed L_β to a structure $\langle L_{\beta^*}, M \rangle$ (so this collapse is "smaller" in case that $\beta > \sigma 1 \mathrm{cf}\, \beta > \beta^*$) such that for any set $A \subseteq L_{\beta^*}$ we have that A is $\Sigma_1 \langle L_{\beta^*}, M \rangle$ iff A is $\Sigma_2\ L_\beta$ [1] .

The structure $\langle L_{\beta^*}, M\rangle$ is in general not admissible but nevertheless one can handle a $\Sigma_1 \langle L_{\beta^*}, M\rangle$ set in many situations better than a Σ_2 L_β set (see the proof of the Uniformization Theorem [1]). The advantage of the collapse $\mathfrak{A}$ is different : It doesn't save a quantifier but makes it possible to reduce questions about β-degrees to questions about $\mathfrak{A}$-degrees which are easier to solve due to the admissibility of $\mathfrak{A}$. In particular a combinatorial argument shows that the structure of β-recursive degrees with $\leq_\beta$ and the structure of $\mathfrak{A}$-r.e. degrees with $\leq_{\mathfrak{A}}$ are isomorphic. Since there exist incomparable r.e. degrees in $\mathfrak{A}$ (one has to use a variation of the Sacks-Simpson proof since $\mathfrak{A}$ is an admissible structure with an additional predicate , see [8] ,[10]) we get incomparable β-recursive degrees for weakly inadmissible β . Much less is known about the strongly inadmissible case. The rest of this paper is devoted to a proof of the following result:

<u>Theorem :</u> Assume that β is strongly inadmissible and β^* is a regular β-cardinal. Then there exist incomparable β-recursive degrees.

The <u>proof</u> is based on arguments that have been introduced into β-recursion theory by S.Friedman in order to construct incomparable β-r.e. degrees for the same class of β [2],[5] . These arguments are of special interest in the context of this paper since their heart is a <u>combinatorial principle</u> which is very close to <u>$\diamond$</u> [1] (Jensen used the validity of $\diamond$ in L in order to disprove the Souslin Hypothesis in L).

It seems hopeless to construct directly Δ_1 L_β sets of incomparable degree. Therefore we use a little trick. We construct β-r.e. sets A and B in $\beta^* \cdot \varkappa$ steps (i.e. we run $\varkappa$ times through β^*) where <u>$\varkappa$ is in the following the β-cardinal $\sigma 1 \mathrm{cf}\, \beta$</u> . Define

$\langle \delta, x\rangle \in A_{rec}$ $:\Leftrightarrow$ (x is put into A before step $\beta^* \cdot (\delta+1)$) .

Since A_{rec} is β-recursive and since one can see immediately that $A \leq_{w\beta} A_{rec}$ we construct A and B in such a way that $\neg A \leq_{w\beta} B_{rec}$ and $\neg B \leq_{w\beta} A_{rec}$ hold where B_{rec} is defined like A_{rec}. The relation " $\leq_{w\beta}$ " -weakly β-reducible to- is defined like " $\leq_\beta$ " on the second page but with the sets K in the definition restricted to one element sets $\{x\}$. Since we have that $D_1 \leq_{w\beta} D_2$ and $D_2 \leq_\beta D_3$ implies $D_1 \leq_{w\beta} D_3$ for any sets D_1, D_2, D_3 the β-recursive sets A_{rec}, B_{rec} will then be incomparable with respect to $\leq_\beta$.

We fix as in [5] β-finite sequences $\langle S_\gamma \mid \gamma < \beta^* \rangle$:
If there is a β-cardinal ϱ such that $\beta^* = \varrho^+$ (i.e. β^* is the next β-cardinal after ϱ) define

$$S_\gamma := \mathcal{P}(\gamma) \cap L_{\hat{\gamma}} \quad \text{where } \hat{\gamma} > \gamma \text{ is minimal such that } L_{\hat{\gamma}} \models [\operatorname{card}(\gamma) \leq \varrho] .$$

Otherwise define $S_\gamma := \mathcal{P}(\gamma) \cap L_{\beta^*}$.

Then in both cases the β-cardinality of S_γ is less than β^* for every γ .

If we write some $\sigma < \beta^* \cdot \kappa$ in the form $\sigma = \beta^* \cdot \delta + \gamma$ we mean that $\gamma < \beta^*$ and this γ is called the <u>stage</u> of step σ .

A_σ will be the set of elements which have been put into A before step σ .

The following requirements will be satisfied during the construction :

R_e^A : $\neg A \leq^e_{w\beta} B_{rec}$

T_e^A : $K_e \subseteq A \Rightarrow \exists \delta < \kappa \, (K_e - A_{\beta^* \cdot \delta}$ is bounded below β^*)

S_e^A : $K_e \subseteq \kappa \times \beta^* - A_{rec} \Rightarrow \exists \delta < \kappa (K_e - \delta \times \beta^*$ is bounded below β^*)

and analogous requirements R_e^B, T_e^B, S_e^B where the roles of A and B are interchanged (K_e is an abbreviation for $K(e)$ where K is some fixed β-recursive function from β onto L_β). Observe that if $\beta^* - A$ is unbounded in β^* and all requirements S_e^A are satisfied then A is simple but the converse doesn't hold.

The set M_σ is the set of those requirements which are considered at step $\sigma = \beta^* \cdot \delta + \gamma$ of the construction (f is a fixed β-recursive projection of β into β^* ; q is a fixed β-recursive cofinal function from κ into β ; we write $f^{q(\delta)}(e)\downarrow$ if $L_{q(\delta)} \models [\, f(e)\downarrow]$):

M_σ is the set of all S_e^A, S_e^B, T_e^A, T_e^B, $\langle g, R_e^A \rangle$, $\langle g, R_e^B \rangle$ such that $f^{q(\delta)}(e)\downarrow$, $f^{q(\delta)}(e) < \gamma$ and $g \in S_\gamma$.

We fix a well-ordering $<_\sigma$ of every M_σ in such a way that the map $\sigma \mapsto <_\sigma$ is β-recursive.

<u>Construction :</u> <u>Step $\sigma = \beta^* \cdot \delta + \gamma$</u> : We will only describe the A part of the construction, the B part is analogously.

If stage γ was cancelled at some step $\sigma' < \sigma$ we do nothing and proceed to the next step.

If $\gamma = 0$ we first put all y into A such that " $\langle \delta, y \rangle \in A_{rec}$" was promised at some step $\sigma' = \beta^* \cdot \delta' + \gamma' < \sigma$ and stage γ' was not cancelled before step σ .

The following will be done then at step σ (for $\gamma = 0$ and $\gamma > 0$) :
We run through the set M_σ according to the well-ordering $<_\sigma$ and make attempts for the elements if possible as it is described in the following :

Assume thet $Q \in M_\sigma$ is considered next during this run through M_σ . If an attempt was made for Q at some step $\tilde{\sigma} = \beta^* \cdot \tilde{\delta} + \gamma < \sigma$ we go to the next requirement in M_σ . Otherwise the following will be done : We write $\tilde{A}$ for the set of those elements which have been put into A until then (i.e. at some step $\sigma' < \sigma$ or during an attempt at step σ for some $Q' \in M_\sigma$ such that $Q' <_\sigma Q$) ; $\tilde{B}$ is analogously defined.

We further define :

$\hat{P}^A := \{ <\delta',y> \mid y \geq \gamma \wedge ((y \in \tilde{A}$ and y was put into A at some step $\sigma' = \beta^* \cdot \delta' + \gamma'$ with $\sigma' \leq \sigma$) ($y \notin \tilde{A}$ and it was promised at some step $\tilde{\sigma} = \beta^* \cdot \tilde{\delta} + \gamma$ (!) with $\tilde{\sigma} \leq \sigma$ to satisfy "$<\delta',y> \in A_{rec}$"))

and

$P^A := \{ <\tilde{\delta},y> \mid \tilde{\delta} < \kappa \wedge \exists <\delta',y> \in \hat{P}^A (\delta' \leq \tilde{\delta}) \}$ and

$N^A := \{ <\tilde{\delta},y> \mid y \geq \gamma \wedge ((\tilde{\delta} < \delta \wedge y \notin A_{\beta^* \cdot (\delta+1)}) \vee (\delta \leq \tilde{\delta} < \kappa \wedge$ ($y \notin \tilde{A}$ and at some step $\tilde{\sigma} = \beta^* \cdot \tilde{\delta} + \gamma$ with $\tilde{\sigma} \leq \sigma$ it was promised to satisfy "$<\delta',y> \notin A_{rec}$" for some $\delta' \geq \tilde{\delta}$))) $\}$.

The sets P^B, P^B, N^B are defined analogously.

For sets $M_1, M_2 \subseteq \kappa \times \beta^*$ we say that M_1, M_2 are <u>consistent</u> $:\Leftrightarrow \neg \exists \delta_1 \delta_2 y (\delta_1 \leq \delta_2 \wedge <\delta_1,y> \in M_1 \wedge <\delta_2,y> \in M_2)$.

Observe that we are using a β-finite pairing function $< , > :$ $\beta^* \times \beta^* \rightarrow \beta^*$ (with associated projections π_1, π_r) such that for any sets $M_1, M_2 \subseteq \beta^*$ we have that $M_1 \times M_2$ is bounded below β^* iff M_1 and M_2 are bounded below β^*.

We have now fixed the notation at the point of the construction where we start to consider the requirement $Q \in M$.

If <u>$Q \equiv <g,R_e^A>$</u> we look then whether a tripel $<x,K,H>$ exists such that $x > \gamma \wedge \pi_r[K] \cap \gamma+1 = \emptyset \wedge \pi_r[H] \cap \gamma = \emptyset \wedge x \notin \tilde{A} \wedge$ $K,H \in L_{\beta^*} \wedge K,H \subseteq \kappa \times \beta^* \wedge P^B \cup K, N^B \cup H$ are consistent $\{<\delta,x>\}$, N^A are consistent $\wedge \exists H_1 H_2 \in L_{q(\delta)} ((L_{q(\delta)} \models$ $[<\{x\},H_1,H_2,1> \in W_e]) \wedge H_1 \subseteq P^B \cup K \cup g \wedge H_2 \subseteq N^B \cup H \cup (\kappa \times \gamma - g))$.

If such a tripel doesn't exist we go to the next requirement in M_σ. Otherwise we choose the existing tripel $<x,K,H>$ minimal with respect to $<_\beta$ and make an attempt for $<g,R_e^A>$ at step σ with $<x,K,H>$. We put then immediately x into A and every y such that $<\delta,y> \in K$ into B . For $<\delta',y> \in K$ with $\delta' > \delta$ we further promise to satisfy "$<\delta',y> \in B_{rec}$" . For $<\delta',y> \in H$ with $\delta' \geq \delta$ we promise to satisfy "$<\delta',y> \notin B_{rec}$" .

If $Q \equiv T_e^A$ or $Q \equiv S_e^A$ we do nothing if $K^{q(\delta)}(e)\uparrow$.

Otherwise if $\underline{Q \equiv T_e^A}$ we look whether some $y \geq \gamma$ exists such that $y \in K^{q(\delta)}(e) \wedge y \notin \tilde{A} \wedge P^A$, H consistent with $H := \varkappa \times \{y\}$. If such a y exists we make an attempt for T_e^A and take the least such y and promise to satisfy " $<\delta', y> \notin A_{rec}$" for all $\delta' \geq \delta$.

For $\underline{Q \equiv S_e^A}$ we look whether $y > \gamma$ exists such that $<\delta', y> \in K^{q(\delta)}(e)$ for some $\delta' \geq \delta$ and $\{<\delta, y>\}$, N^A are consistent. If such a y exists we make an attempt for S_e^A and put the least such y into A .

The cases where Q is a requirement $<g, R_e^B>$, T_e^B, S_e^B are treated analogously.

In any case if we make an attempt for Q at step σ and put some y into A (B) or promise to satisfy " $<\delta', y> \in A_{rec}$ (B_{rec})" or " $<\delta', y> \notin A_{rec}$ (B_{rec})" we simultaneously cancel every stage $\gamma' > \gamma$ with $y \geq \gamma'$.

If all requirements $Q \in M_\sigma$ have been considered in the described way we go to the next step.

<u>End of construction.</u>

It is not difficult to see that the construction is β-recursive: By using the definition of $\varkappa$ one can define by recursion a β-recursive function $F : \varkappa \to L_\beta$ such that $F(\delta) = f_\delta$ is for every $\delta < \varkappa$ a β-finite function from β^* into L_{β^*} where $f_\delta(\gamma)$ describes what happens at step $\beta^* \cdot \delta + \gamma$ of the construction for every $\gamma < \beta^*$. One can see that every f_δ is a β-finite function by using the fact that β^* is a regular β-cardinal. It follows immediately from these considerations that A, B are β-r.e. and that A_{rec} , B_{rec} are β-recursive sets.

The construction is a variation of Friedman's construction [5] and in order to show that this construction works as it is supposed to do a fixpoint argument is applied. These kind of arguments do not occur in ORT and are an ingredient from set theory. The difference to ORT will become clear in the last Lemma where one uses not an induction on the priority. Fixpoint arguments are used as well in more advanced parts of α-recursion theory (e.g. Density Theorem [11] , existence of minimal pairs [12] , existence of high degrees [7]).

Here we consider fixpoints of Σ_1 skolem hulls :
For the parameter $q \in L_\beta$ of the preceding construction and the more interesting case where β^* is a successor β-cardinal ϱ^+ define

$$F := \{ \gamma < \beta^* \mid \gamma = ((\ \Sigma_1 \text{ skolem hull of } \gamma \cup \varrho+1 \cup \{q\} \text{ in } L_\beta) \cap \beta^*) \ .$$

Then F is unbounded in β^* since

$$\gamma_\delta := ((\ \Sigma_1 \text{ skolem hull of } \delta \cup \varrho+1 \cup \{q\}\ ) \cap \beta^*)$$

is an element of F for every $\delta < \beta^*$.

The effective version $\langle S_\gamma \mid \gamma < \beta^* \rangle$ of $\diamondsuit'$ is used in order to overcome the difficulty that we don't have in strongly inadmissible β that for every $\gamma < \beta^*$ a step $\sigma_\gamma < \beta^* \cdot \kappa$ exists such that $A \cap \gamma = A_\sigma$. Therefore one tries to satisfy the requirements R_e^A, R_e^B for every "guess" g concerning $A \cap \gamma$ and $B \cap \gamma$. $\diamondsuit'$ gives for every γ a small set S_γ of guesses g and S_γ is defined in such a way that $A \cap \gamma \in S_\gamma$ and $B \cap \gamma \in S_\gamma$ for every $\gamma \in F$ (this is shown by considering the transitive collapse of a suitable Σ_1 skolem hull) .

In the construction here we use $\diamondsuit'$ in order to guess not only which elements will finally be put into A respectively B but as well when these elements will be put into A respectively B , i.e.

we guess at $A_{rec} \cap \gamma$ and $B_{rec} \cap \gamma$.

Observe that during the construction one doesn't know which γ are elements of F . Thus one treats every $\gamma < \beta^*$ as if it would be an element of F .

<u>Lemma i) :</u> Let M be the set of those $\gamma < \beta^*$ such that

a) stage γ is never cancelled and

b) $\kappa < \gamma \wedge \kappa \times \gamma \subseteq \gamma$ and

c) $A_{rec} \cap \gamma \in S_\gamma \wedge B_{rec} \cap \gamma \in S_\gamma$.

Then M is unbounded in β^*.

<u>Proof:</u>

<u>Case 1 :</u> $\beta^* = \varrho^+$ (i.e. ϱ and β^* are successive β-cardinals)

It is easy to check that $F - (\varrho + 1)$ is a subset of M .

<u>Case 2 :</u> β^* is a limit β-cardinal

For this less serious case one constructs for any given γ_0 with $\kappa < \gamma_0 < \beta^*$ a continuous increasing sequence $\langle \gamma_\tau \mid \tau < \kappa \rangle$ as follows

$\gamma_{\tau+1} > \gamma_\tau$ is minimal such that at steps $\sigma = \beta^* \cdot \delta + \gamma$ with $\gamma < \gamma_\tau$ and $\delta < \tau$ only stages $\tilde{\gamma} < \gamma_{\tau+1}$ are cancelled and such that $\kappa \times \gamma_\tau \subseteq \gamma_{\tau+1}$.

Since $\sigma 1 cf^{L_\beta}(\beta^*) = \beta^*$ holds [5] we have that $\gamma := \lim_{\tau < \kappa} \gamma_\tau$ is an ordinal less than β^* and it is easy to see that $\gamma \in M$.

<u>Lemma ii) :</u> Assume that at some step $\sigma = \beta^* \cdot \delta + \gamma$ with $\gamma \in M$ of the construction it is promised at an attempt for some requirement $Q \in M_\sigma$ to satisfy $\langle \delta', y \rangle \in A_{rec}$ or $\langle \delta', y \rangle \notin A_{rec}$. Then we have in fact that $\langle \delta', y \rangle \in A_{rec}$ respectively $\langle \delta', y \rangle \notin A_{rec}$. (analoguously for B_{rec}).

Proof : The case " $<\delta',y> \in A_{rec}$" is trivial. For the case " $<\delta',y> \notin A_{rec}$" assume that $Q \equiv <g,R_e{}^B>$ (the case $Q \equiv T_e{}^A$ is treated similarly) and that $\tilde{A}$, P^A, N^A are the sets which are defined immediately before the attempt with tripel $<x,K,H>$ is made for this $Q \in M_\sigma$. We have then $y \geq \gamma$, $y \notin \tilde{A}$, $\delta' \geq \delta$ and $<\delta',y> \in H$.

Since $P^A \cup K$, H are consistent it can't be that y is put into A because of this attempt for Q .

Further the cancelling of stages during the construction makes sure that $<\delta',y> \in A_{rec}$ can only happen because of an attempt which is made for a requirement $\hat{Q}$ after the attempt for Q at some step $\hat{\sigma} = \beta^* \cdot \hat{\delta} + \gamma$ with the same stage γ and $\hat{\delta} \leq \delta'$. One has then $<\delta',y> \in \hat{N}^A$ where $\hat{N}^A$ is the set N^A which is defined immediately before this attempt for $\hat{Q}$. Therefore at step $\hat{\sigma}$ y is neither put into A at once nor is a promise made to satisfy $<\delta'',y> \in A_{rec}$ for some $\delta'' \leq \delta'$ because of the consistency condition for this attempt.

Lemma iii) :

a) $\beta^* - A$ and $\beta^* - B$ are unbounded in β^*

b) $K \subseteq A \Rightarrow \exists \delta < \varkappa$ ($K - A_{\beta^* \cdot \delta}$ is bounded below β^*)
(analogous for B)

c) $K \subseteq \varkappa \times \beta^* - A_{rec} \Rightarrow \exists \delta < \varkappa$ ($K - \delta \times \beta^*$ is bounded below β^*)
(analogous for B)

d) $\neg A \leq_{w\beta} B_{rec}$ and $\neg B \leq_{w\beta} A_{rec}$.

Proof : The proof of a) is trivial since $M \subseteq \beta^* - A$ and $M \subseteq \beta^* - B$.

We need b) and c) only in order to make the proof of d) possible and b), c) are proved similarly as d).

In order to prove d) assume for a contradiction that $A \leq^e_{w\beta} B_{rec}$. By Lemma i) there exists $\gamma_0 \in M$ such that $f(e) < \gamma_0$. We have then that $g_0 := B_{rec} \cap \gamma_0 \in S_{\gamma_0}$. We want to show that an attempt for $\langle g_0, R_e^A \rangle$ will be made at some step $\sigma = \beta^* \cdot \delta + \gamma_0$.

Take some $\gamma_1 \in M$ such that $\gamma_1 > \gamma_0$ and some $x \in \beta^* - A$ such that $x > \gamma_1$. Choose $\delta_0 \in \varkappa$ such that (use b) and c)) :

$$\exists H_1 H_2 \in L_{q(\delta_0)}((L_{q(\delta_0)} \models [\langle \{x\}, H_1, H_2, 1 \rangle \in W_e])$$
$$\wedge\ H_1 \subseteq B_{rec} \wedge H_2 \subseteq \varkappa \times \beta^* - B_{rec}$$
$$\wedge\ (\pi_r [H_1] - B_{\beta^* \cdot \delta_0} \text{ and } H_2 - \delta_0 \times \beta^* \text{ are bounded below } \beta^*.$$

If there exists no $\delta < \delta_0$ such that an attempt is made for $\langle g_0, R_e^A \rangle$ at step $\beta^* \cdot \delta + \gamma_0$ then an attempt for $\langle g_0, R_e^A \rangle$ will be made at step $\sigma_0 := \beta^* \cdot \delta_0 + \gamma_0$.

The tripel $\langle x, K, H \rangle$ with $K := B_{rec} \cap \varkappa \times (\pi_r [H_1] - (B_{\beta^* \cdot \delta_0} \cup \gamma_0))$ and $H := H_2 - (\delta_0 \times \beta^* \cup \varkappa \times \gamma_0)$ has all the properties (except perhaps minimality) which are required in order to make an attempt for $\langle g_0, R_e^A \rangle$ at step σ_0 with $\langle x, K, H \rangle$. In particular K is β-finite (since $\pi_r [H_1] - B_{\beta^* \cdot \delta_0}$ is bounded below β^* and $\beta^* = \varrho_{1,\beta}$) and we have that $P^B \cup K$, $N^B \cup H$ are consistent since otherwise we would get a contradiction to $K \subseteq B_{rec} \wedge H \subseteq \varkappa \times \beta^* - B_{rec}$ by using Lemma ii).

Thus we can be sure that an attempt is made for $\langle g_0, R_e^A \rangle$ at some step $\sigma = \beta^* \cdot \delta + \gamma_0$ with a tripel $\langle x, K, H \rangle$. It follows then from Lemma ii) that

$$\exists H_1 H_2 (\langle \{x\}, H_1, H_2, 1 \rangle \in W_e \wedge H_1 \subseteq B_{rec} \wedge H_2 \subseteq \varkappa \times \beta^* - B_{rec})$$

and since x is put into A at step σ we have got a counterexample to $A \leq^e_{w\beta} B_{rec}$.

This finishes the proof of the Theorem.

Remarks :

1) The constructed sets A , B have the same properties as in [5] : They are weakly tame r.e. (i.e. b) of Lemma iii) holds) and incomparable with respect to " $\leq_{w\beta}$ " .

2) Before the Theorem we mentioned already the argument for the strongly inadmissible case in Lemma 12, p.93 in [1] , which is one step in the proof of the Uniformization Theorem. This simple combinatorial argument was in fact the key to our proof of the preceding Theorem and it can be isolated as follows :

Assume that β is strongly inadmissible and $q : \sigma 1 cf \beta \to \beta$ is a cofinal β-recursive function. Further assume that the set $A \subseteq L_{\beta^*}$ is defined by the Σ_1 formula Ψ over L_β and define the set $A_{rec} \subseteq L_{\beta^*}$ by $\langle \delta , x \rangle \in A_{rec} \iff L_{q(\delta)} \models \Psi(x)$.
Then we have $A \leq_{\beta^*} A_{rec}$ and the set A_{rec} is β-recursive.

The proof of this fact is trivial : One just observes that for $K \in L_{\beta^*}$ such that $K \subseteq A$ we have that $A_{rec} \cap \sigma 1 cf \beta \times K \in L_{\beta^*}$.

The fact holds as well for structures $\mathfrak{B} \equiv \langle L_\beta , B \rangle$ (B regular over L_β) with $\rho^{\mathfrak{B}}_{1,\beta}$ instead of β^* .

Besides the application of this fact in the proof of the Uniformization Theorem one can use it in order to show that for every strongly inadmissible β and weakly tame r.e. $A \subseteq L_{\beta^*}$ (see [5]) we have that $A \leq_\beta A_{rec}$.

There is another application in α-recursion theory which gives a rather unexpected result :

Assume that $\sigma 2 cf \alpha < \sigma 2 p \alpha = \alpha$ (e.g. $\alpha = \aleph^L_\omega$). Then there is a Δ_2 L_α set of degree 0'' .

(See [7] for other results about Δ_2 L_α degrees .)

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