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ON MINIMAL PAIRS AND MINIMAL DEGREES IN HIGHER RECURSION THEORY

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Priority arguments are the hallmark of recursion theory, so it is natural to ask how far they can be generalized to higher recursion theory. As it turns out, many priority constructions of recursion theory on ω can in fact be carried out in α-recursion theory for admissible α and a few of them generalize to recursion in higher types as well.

The common method of generalization is, to avoid crude assumptions about the underlying domain which are true for ω but not for all admissible α (for example full replacement holds in L_ω) by using a more sophisticated construction, for which weaker closure properties of the domain are in fact enough. So these generalizations are neither trivial nor uniform.

The two constructions which are considered in this paper are of special interest because they obstinately resist generalization to all admissible α. Lerman and Sacks constructed in [6] α-r.e. non α-recursive sets A, B which form a minimal pair for those admissible α, which are not refractory (i.e. each $C \subsetneqq \alpha$ with $C \leqq_\alpha A$ and $C \leqq_\alpha B$ is in fact α-recursive). They called α refractory, if $p2\alpha = gc\alpha < tp2\alpha \leqq \alpha$, recently Shore covered the case $tp\,2\alpha = \alpha$ (see [13]). We show here, that using an appropriate choice of priorities, the minimal pair can be made hyperregular. We further show that the Lerman-Sacks construction works for the open case $gc\alpha < tp2\alpha < \alpha$ if L_α is the limit of Σ_1 substructures L_β with $p2\beta^{L_\beta} = \beta$. The reasons for the interest in hyperregularity are philosophical and technical due to the fact that, for a hyperregular α-r.e. set A, the following notions are the same: $\leqq_\alpha A$ (α-rec. in A), $\leqq_{w\alpha} A$ (weakly α-rec. in A), $\leqq_{c\alpha} A$ (α-finitely calculable from A), $\Sigma_1\langle L_\alpha A\rangle$ (see [9] for details). We apply the construction of a hyperregular minimal pair to construct α-r.e. sets A, B, C for the same α such that C is not α-rec. and a nontrivial glb of $\{A, B\}$, generalizing a construction of Lachlan [5] for ω. Applying Harrington's translation, we further get a minimal pair for recursion in higher types and a nonzero branching functional. Concerning minimal degrees Shore constructed [12] a set $B \leqq_\alpha 0'$ for Σ_2-admissible α such that no $C \subsetneqq \alpha$ exists with $0 <_\alpha C <_\alpha B$. We generalize his construction to yield minimal degrees for those admissible α where $p\,2\,\alpha \leqq cf\,2\alpha$ which includes the case $s3p(\alpha) \leqq cf\,2\alpha$.

Finally we apply a technique from [8] and show that both constructions succeed for some α that are not admissible. Thus one is in the situation where admissibility is neither sufficient (on the basis of the known constructions) nor necessary for these constructions. So it is tempting to assume, that the appropriate condition for α to yield a suitable α-recursion theory is formulated in terms of projecta and cofinalities, rather than in terms of mere admissibility.

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§ 1. Hyperregular Minimal Pairs

α is always assumed to be admissible.

The notations for projecta and cofinalities are the same ones used in Lerman-Sacks [6]. We assume that the reader is familiar with the basic notions and facts of α-recursion theory (see Shore [15] for an excellent introduction and bibliography). Partial function $[e]^A$ weakly recursive in A with index e are defined as usual:

$$[e]^A(\gamma)=\delta \leftrightarrow \exists K,\ H\in L_\alpha(\langle\gamma,\delta,K,H\rangle\in W_e \wedge K\subseteqq A \wedge H\subseteqq \alpha - A),$$

writing $[e]^A(\gamma)\uparrow$ if several values δ occur [where $(W_e)_{e\in\alpha}$ is a simultaneous enumeration of the α-r.e. sets]. One needs further an approximation $[e]^A_\sigma$ of $[e]^A$, using only computations $\langle\gamma,\delta,K,H\rangle\in L_\sigma$ which are enumerated in W_e before stage σ, such that $[e]^A(\gamma)\cong\lim_{\sigma\to\alpha}[e]^{A^\sigma}_\sigma(\gamma)$ for any enumeration of a regular α-r.e. set A and any γ (we always write A^σ for the part of A enumerated before σ). It is tempting to define $[e]^A_\sigma$ exactly as $[e]^A$ by simply replacing W_e by $W_e^\sigma\cap L_\sigma$, writing again $[e]^A_\sigma(\gamma)\uparrow$ if several values δ occur. However the approximation might fail then, because for unboundedly many $\sigma[e]^{A^\sigma}_\sigma(\gamma)$ might have several values, and therefore $[e]^{A^\sigma}_\sigma(\gamma)\uparrow$ even if $[e]^A(\gamma)$ is well defined. We take therefore $[e]^A_\sigma(\gamma)\downarrow$ even if several values occur, in which case we search for the least $\tau\leqq\sigma$ such that $[e]^A_\tau(\gamma)$ has a value and set $[e]^A_\sigma(\gamma)$ equal to the least of these values at stage τ. We call the neighborhoods K and H, leading to this least value at stage τ, respectively the positive and the negative neighborhood of the computation $[e]^A_\sigma(\gamma)$. It is easy to verify the desired approximation property for this definition.

The strategy of Lerman-Sacks [6] to construct a minimal pair A, B (going back to Lachlan [5]) is roughly as follows: A and B are simultaneously enumerated. During the construction one tries to satisfy the requirements $\{e\}\neq c_A$ and $\{e\}\neq c_B$ in order to make A, B non-α-recursive [$(\{e\})_{e\in\alpha}$ is an enumeration of the partial α-recursive functions, c_A and c_B are the characteristic functions of A and B respectively, equalling 0 for elements in the set]. These requirements, which are abbreviated R^A_e, R^B_e, are treated according to their priority. Let us assume that at stage σ all requirements of higher priority than a certain R^A_e are discharged (i.e. they don't receive attention at a stage $\tau\geqq\sigma$). We consider then the computation $\{e\}(\sigma)$ and try to make $\{e\}(\sigma)\neq c_A(\sigma)$ by putting σ into A if $\{e\}(\sigma)\cong 1$ and leaving it out of A otherwise. The problem is that the computation of $\{e\}(\sigma)$ might be very long [we write $\{e\}^\tau(\sigma)\cong\delta$ iff the computation $\{e\}(\sigma)$ has length $<\tau$] or might not converge.

So in general we can't do any more at stage σ than appoint σ as an unrealized follower for R_e^A and wait until a stage $\tau > \sigma$, where $\{e\}^\tau(\sigma)\downarrow$ (σ is a follower of R_e^A, if we try to satisfy $\{e\} \neq c_A$ at the point σ, σ is unrealized as long as we don't know whether it should be in or out of A for this purpose). If $\{e\}^\tau(\sigma)\downarrow$, σ is a realized follower at τ and we know then at stage τ what we have to do with σ. Nevertheless we can't always do what we want, because we have to satisfy further requirements Q_j, which insure that in the case where $f=[j_0]^A=[j_1]^B$ is a total function, f is in fact α-recursive. This is essentially done in the following way. If at stage ϱ an equality $[j_0]_\varrho^{A_\varrho}(\delta)=[j_1]_\varrho^{B_\varrho}(\delta)$ occurs, we try to preserve these two computations by avoiding putting elements in A or B, which are in the negative neighborhood of the respective computation. If we can do this, at least from some stage ϱ_0 on, we can in fact compute f α-recursively: we just take the value of the first equality with argument δ, occuring after ϱ_0, to be the value $f(\delta)$. However we can't preserve all these equalities, if we want to satisfy all requirements R_e^A, R_e^B as well. So we are a little more liberal and allow the computation on one side of the equality to be destroyed in order to satisfy a R_e^A or R_e^B. We can afford to do so because the remaining computation still preserves the right value and, if it is really the case that $[j_0]^A(\delta)=[j_1]^B(\delta)\downarrow$, the destroyed side of the equality will be reestablished anyway (due to the approximation property). But we have to enforce strictly, that the remaining computation not be destroyed by any requirement of lower priority than Q_j, until the other side of the computation is reestablished. Therefore a follower σ of a requirement R_e^A of lower priority, which would destroy the remaining computation by getting into A, is put on a waiting list (we say: σ is associated with Q_j), where σ remains until the equality is established again and σ is therefore allowed to destroy one of them. Of course, σ may have a stroke of bad luck and the equality may never be reestablished (it may happen that $[j_0]^A(\delta)\uparrow$ or $[j_1]^B(\delta)\uparrow$) and thus it may end up waiting forever. In order to satisfy R_e^A anyway, we always appoint at a stage a new unrealized follower of R_e^A, if all the other followers of R_e^A are associated with some Q_j, hoping that one of them eventually comes through.

Requirements H_e^A, H_e^B, which make the constructed sets hyperregular, are well known. A is not hyperregular, iff there is an e and a $\delta<\alpha$ such that $[e]^A$ maps δ unboundedly into α. In order to avoid this, computations $[e]_\tau^{A^\tau}(\gamma)$ for $\gamma<\delta$ are preserved from some stage on by cancelling followers of requirements R_e^A of lower priority which might destroy the computation later by coming into A. $[e]^A$ can then be computed α-recursively as the previous function f and is therefore bounded. Adding these requirements to the minimal pair construction, the following difficulty occurs: One shows in the Lerman-Sacks construction inductively that requirements R_e from a short priority list of requirements of length less than κ (where κ is a regular α-cardinal) receive attention less than κ times (because their followers only run through a short waiting list of Q_j's of length $<\kappa$). If we now place the H_e between the R_e on the priority list, the H_e may disrupt this process, because they may receive attention κ many times (if $\delta=\kappa$ for the δ above). We would nevertheless succeed, if we had no more than $cf\,2\alpha$ many H_e's on the priority list, arguing as before for R_e's between two H_{e_1}, H_{e_2} and using the definition of $cf\,2\alpha$ for limit points. But we can't expect to always be able to press the H_e into such a short list. We place therefore $cf2\alpha$ many blocks of H_e's rather than single H_e's on the priority list, observing that it is essentially not more difficult to satisfy a whole block of H_e's than it is to satisfy a single H_e.

It is easy, to verify that $cf2\alpha = cf2(tp2\alpha)$ see Lemma 2.15 in Chong-Lerman [1]). Further the cofinal Σ_2 function $q: cf2\alpha \to tp2\alpha$ can be defined as strictly increasing and continuous, such that for $\gamma < \beta < \mathrm{cf}2\alpha$ $\forall\delta(q(\gamma)+\delta = q(\gamma+1) \to q(\beta)+\delta \leqq q(\beta+1))$. According to Chong-Lerman [1], Lemma 2.17 we have $tp2\alpha = p2\alpha$ or $tp2\alpha = p2\alpha \cdot cf2\alpha$ (observe that this implies that for no admissible α $p2\alpha < gc\alpha < tp2\alpha < \alpha$) so for the latter case we may define $q(\gamma) = p2\alpha \cdot \gamma$. For the case $tp2\alpha = p2\alpha$ we define q such that $q(\beta+1) \geqq q(\beta) + q(\beta)$ for all $\beta < cf2\alpha$. A simple way to see that a q with these properties has a Σ_2-definition is to define q $\mathfrak{B}$-recursive in the structure $\mathfrak{B} = \langle L_\alpha, C\rangle$ with a complete Σ_1 C, which is in general inadmissible, but where a $\mathfrak{B}$-recursive function may be defined by recursion of length $cf2\alpha$ (see [8]). One can further find an α-recursive $q': \alpha \times cf2\alpha \to \alpha$ such that for all τ $q'(\tau, \cdot)$ is increasing and continuous and q is the tame limit of q'.

We define now priorities for the requirements $P_e^0 = R_e^A$, $P_e^1 = P_e^B$, $P_e^2 = H_e^A$, $P_e^3 = H_e^B$. P_e^i gets the priority $p(P_e^i) := p(\omega e + i)$ where $p: \alpha \to tp2\alpha$ accomplishes the blocking as follows: Let $f: \alpha \to tp2\alpha$ be a one-one tame Σ_2 function with an α-recursive tame approximation f' and q as before. For $f(\omega e + n) = q(\gamma) + \delta < q(\gamma+1)$ we set

$$p(\omega e + n) = \begin{cases} q(2\gamma) + \delta, & \text{if } n < 2 \\ q(2\gamma + n) + \delta, & \text{if } n \geqq 2. \end{cases}$$

p is obviously Σ_2 and one-one. An α-recursive tame approximation $p': \alpha \times \alpha \to \alpha$ of p is defined by setting

$$p'(\tau, \omega e + n) = \begin{cases} \sup\limits_{\gamma < cf2\alpha} q'(\tau, \gamma) & \text{if } f'(\tau, \omega e + n) \geqq \sup\limits_{\gamma < cf2\alpha} q'(\tau, \gamma) \\ q'(\tau, 2\gamma) + \delta & \text{if } q'(\tau, \gamma) + \delta = f'(\tau, \omega e + n) < q'(\tau, \gamma + 1) \text{ and } n < 2 \\ q'(\tau, 2\gamma + 1) + \delta & \text{if } n \geqq 2. \end{cases}$$

According to Lerman Sacks an α-recursive $q': \alpha \times \alpha \to \alpha$ is a tame approximation of $q: \alpha \to \beta$ if

$$(*) \quad \forall z < \beta \exists \sigma \forall x \forall \tau \geqq \sigma ((q(x) \leqq z \to q'(\tau, x) = q(x)) \wedge (q(x) > z \to q'(\tau, x) > zz)).$$

If q is one-one then we can always change q' slightly so that in addition $q'(\tau, \cdot)$ is one-one for all τ.

We define the blocks B_δ by $B_\delta = \{\gamma \mid q(\delta) \leqq \gamma < q(\delta+1)\}$ for $\delta < cf2\alpha$. Speaking of a block B_δ of requirements we mean the set of requirements P_e^i with $p(P_e^i) \in B_\delta$. By our definition this block of requirements consists only of R_e for $\delta = 2\gamma$ and only of H_e for $\delta = 2\gamma + 1$.

The requirements H_e^A assure that $[e]^A \upharpoonright h_A(e)$ is bounded, if it is total on $h_A(e)$. $h_A: \alpha \to \alpha$ and a tame approximation $h'_A: \alpha \times \alpha \to \alpha$ are defined by:

$$h_A(e) = h'_A(\tau, e) = e \quad \text{for all} \quad \tau, e \quad \text{if} \quad gc\alpha = \alpha.$$

If $gc\alpha=\alpha^*<\alpha$ and $gc\alpha$ is a singular α-cardinal we take an increasing cofinal Σ_2 function $r:cf2\alpha\to\alpha^*$ [which exists because $cf2\alpha=cf2(\alpha^*)$] with an α-recursive tame approximation $r':\alpha\times cf2\alpha\to\alpha$. For $q(\delta)\leqq p(\omega e+2)<q(\delta+1)$ we then set $h_A(e)=r(\delta)$ and

$$h'_A(\tau,e)=\begin{cases}0 & \text{if } p'(\tau,\omega e+2)\geqq \sup\limits_{\gamma<cf2\alpha} q'(\tau,\gamma)\\ r'(\tau,\delta) & \text{if } q'(\tau,\delta)\leqq p'(\tau,\omega e+2)<q'(\tau,\delta+1).\end{cases}$$

For the other cases h_A and h'_A are constant with value $gc\alpha$. h_B and h'_B are defined analogously.

Theorem 1. Assume that either $\neg gc\alpha<tp2\alpha<\alpha$ or $gc\alpha<tp2\alpha<\alpha$ and α satisfies the following property (L): $\sup\{\beta \mid L_B\prec_{\Sigma_1}L_\alpha\wedge p2\beta^{L_\beta}=\beta\}=\alpha$. Then there exist α-r.e. hyperregular sets A,B such that A,B are not α-recursive and that every $C\subseteqq\alpha$ with $C\leqq_{w\alpha}A$ and $C\leqq_{w\alpha}B$ is α-recursive.

Remark. In the case $gc\alpha<tp2\alpha<\alpha$ we always have $\alpha^*=\alpha$ and therefore $\sup\{\beta \mid L_\beta\prec_{\Sigma_1}L_\alpha\}=\alpha$. An example for α with $gc\alpha<tp2\alpha<\alpha$ and property (L) is the following: Let $(L_{\beta_n})_{n\in\omega}$ be the first ω Σ_1 substructures of $L_{\aleph_2^L}$ after $L_{\aleph_1^L}$ with $p2\beta_n^{L_{\beta_n}}=\beta_n$ and define $\alpha=\sup\limits_{n\in\omega}\beta_n$. Such a sequence exists, because if $L_\beta\prec L_{\aleph_2^L}$, then $p2\beta^{L_\beta}=\beta$ (one needs some further basic facts about L, see Devlin [2]). Property (L) is immediate for α and $cf2\alpha=\omega$, $gc\alpha=\aleph_1^L$, therefore $tp2\alpha<\alpha$. *It is not the case that* $tp2\alpha\leqq\aleph_1^L$ (look at the images of the β_n), therefore $gc\alpha<tp2\alpha<\alpha$.

Proof of Theorem 1. We give an exact description of the construction because we need a variation of it for Theorem 2. On the other hand we don't repeat arguments concerning this construction which can be found in Lerman-Sacks [6].
We need not look at the case $p2\alpha=gc\alpha<tp2\alpha=0$, since the minimal pair constructed by Shore is automatically hyperregular.
Let $p_1:\alpha\to p2\alpha$ be Σ_2, one-one and $p'_1:\alpha\times\alpha\to\alpha$ an α-rec. approximation with $\mathrm{Rg}\,p'_1\subseteqq p2\alpha$. $L(\sigma,j):=\mu x(\text{not}\,[(j)_0]_\sigma^{A^\sigma}(x)\cong[(j)_1]_\sigma^{B^\sigma}(x)\downarrow)$. $M(\sigma,j):=\sup\limits_{\tau\leqq\sigma}L(\tau,j)$.

σ satisfies Q_j, if $L(\sigma,j)=M(\sigma,j)$ or Q_j is not persistent at σ $\left(\text{i.e. } p'_1(\sigma,j)\neq\lim\limits_{\tau\to\sigma}p'_1(\tau,j)\right)$.
R_e^A is satisfied at σ, if at some $\tau<\sigma$ a follower p of R_e^A was realized and either $p\in A^\sigma$ or $\{e\}^\tau(p)\cong 0$.
R_e^A requires attention at σ, if R_e^A is not satisfied at σ, $e\leqq\sigma$, and one of the following cases holds
1) R_e^A is not persistent at σ.
2) R_e^A has at σ a realized follower p which is not associated with any Q_j.
3) R_e^A has at σ a follower p which is associated with Q_j and σ satisfies Q_j.
4) R_e^A has at σ an unrealized follower p and $\{e\}^\sigma(p)\downarrow$.
5) R_e^A has no unrealized follower at stage σ.

H_e^A requires attention at σ, if $e \leqq \sigma$ and for some $\gamma < I_e^A(\sigma)$ the negative neighborhood of $[e]_\sigma^{A^\sigma}(\gamma)$ contains a follower of some $R_{e'}^A$ with $p'(\sigma, R_{e'}^A) > p'(\sigma, H_e^A)$;

$$I_e^A(\sigma) := \inf(\{h'_A(\sigma, e)\} \cup \{\delta \mid [e]_\sigma^{A^\sigma}(\delta)\uparrow\}.$$

Construction. Stage $\sigma > 0$: Let P_e^i be the requirement of highest priority at σ which requires attention at σ (i.e. $p'(\sigma, P_e^i)$ is minimal). Cancel all followers of requirements with lower priority at σ.

a) $\boldsymbol{P_e^i = R_e^A}$

If **Case 1)** holds, cancel all followers of R_e^A. If R_e^A requires attention through some follower p according to Cases 2)—4) choose the case such that p is of highest order (p is of higher order than q at σ, if p and q are follower of R_e^A at σ and p was appointed before q was).

Case 2): Well order $\alpha \times \omega$ by $\langle\gamma, n\rangle \prec \langle\delta, m\rangle :\Leftrightarrow n < m$ or $n = m$ and $\gamma < \delta$. Take $\langle\gamma, n\rangle \in p'(\sigma, R_e^A) \times \omega$ minimal with respect to $\prec$ such that $\langle\gamma, n\rangle \succ \langle\delta, m\rangle$ for all $\langle\delta, m\rangle$ through which p was associated with some Q_j before σ and such that $\exists j \leqq \sigma (p'_1(\sigma, j) = \gamma$ and no follower of any requirement is still associated with Q_j at σ). If this $\langle\gamma, n\rangle$, Q_j exists, associate p with Q_j through $\langle\gamma, n\rangle$. Otherwise put p into A and cancel all followers of R_e^A.

Case 3): Cancel the association of p with Q_j. Proceed as in Case 2).

Case 4): p becomes now a realized follower of R_e^A. If not $\{e\}^\sigma(p) \cong 1$ cancel all followers of R_e^A and do nothing further. If $\{e\}^\sigma(p) \cong 1$ proceed as in Case 2).

If R_e^A only requires attention by **Case 5),** appoint σ to be an unrealized follower of R_e^A.

b) $\boldsymbol{P_e^i = H_e^A}$. Do nothing further.

For $B \subseteqq tp2\alpha$ we say that B is discharged at σ, if at no $\tau \geqq \sigma$ a requirement P_e^i with $p'(\tau, P_e^i) \in B$ receives attention.

Lemma 1. Let $\lambda < cf 2\alpha$ be a limit ordinal and assume, that each B_δ with $\delta < \lambda$ is discharged at some σ. Then $B_{<\lambda} := \bigcup_{\delta < \lambda} B_\delta$ is discharged.

Proof. Define a Σ_2 function $g : \lambda \to \alpha$ such that for each $\delta < \lambda$ B_δ is discharged at $g(\delta)$. $\lambda < cf 2\alpha \Rightarrow \mathrm{Rg}(g)$ is bounded by some ϱ. $B_{<\lambda}$ is then *discharged at* ϱ.

In order to prove that each B_δ is discharged, it is now enough to show that for each $\delta < cf 2\alpha$ ($B_{<\delta}$ is discharged $\Rightarrow B_\delta$ is discharged). So assume that $B_{<\delta}$ is discharged at σ_0 and that $(*)$ is satisfied for p, p' and $z = q(\delta + 1)$ at σ_0.

a) $\neg gc\alpha < tp2\alpha < \alpha$ and $\delta = 2\gamma$
B_δ is then a block of requirements R_e. Since no $R_e \in B_\delta$ is injured by any requirement $H_{e'}$ after σ_0 the arguments of Lerman-Sacks [6] show that B_δ is discharged.

b) $\neg gc\alpha < tp2\alpha < \alpha$ and $\delta = 2\gamma + 1$
We may assume that for all $H_e^A, H_e^B \in B_\delta$ and all $\tau \geqq \sigma_0$ $h_A(e) = h'_A(\tau, e), h_B(e) = h'_B(\tau, e)$. Define $T_x := \{\tau \geqq \sigma_0 |$ a requirement P_e^i with $p'(\tau, P_e^i) = x$ receives attention at $\tau\}$ for $x \in B_\delta$. Then $(T_x)_{x \in B_\delta}$ is simultaneously α-r.e. If $H_e^A \in B_\delta$ receives attention at some $\tau \geqq \sigma_0$, for each $\varrho < I_e^A(\tau)$ $[e]_\tau^{A^\tau}(\varrho)\downarrow$ and the negative neighborhood of this computation contains only followers of requirements $R_e^A \in B_{<\delta}$ after stage τ and by the choice of σ_0 the computation is therefore never destroyed. It follows, that if H_e^A receives attention again at some $\tau' > \tau$, we must have $I_e^A(\tau') > I_e^A(\tau)$. By the definition of h_A and h_B we have further that for some constant δ_0 $h_A(e) = \delta_0$ and $h_B(e) = \delta_0$ respectively for all $H_e^A, H_e^B \in B_\delta$. Putting this together we get that T_x has order type $\leqq \delta_0$ for all $x \in B_\delta$.

i) $\alpha^* = gc\alpha < \alpha$ with $gc\alpha$ singular or $gc\alpha = \alpha$.
In these cases we have $\delta_0 < gc\alpha$. Take a regular α-cardinal $\kappa < gc\alpha$ such that $\delta_0 < \kappa$ and $q(\delta + 1) < \kappa$. Applying the Sacks-Simpson-Lemma (Lemma 2.3 in [10]) with respect to κ we get that $\bigcup_{x \in B_\delta} T_x$ is α-finite.

ii) Otherwise. We have then $\delta_0 = gc\alpha$. If $\alpha^* = \alpha$ we argue in the following way: Define $M := \{\langle\beta, x\rangle | \exists \tau \geqq \sigma_0 (H_e \in B_\delta$ receives attention at τ and $p'(\tau, H_e) = x$ and $I_e(\tau) = \beta\}$. M is α-r.e. and $M \subseteqq gc\alpha \times q(\delta \times 1)$, therefore $M \in L_\alpha$. The α-finite function $g: M \to L_\alpha$, which maps $\langle\beta, x\rangle$ onto the τ from the definition of M, has bounded range equal to $\bigcup_{x \in B_\delta} T_x$. If $\alpha^* = gc\alpha$, $gc\alpha$ is regular in this case. Observe that if T_x has order type $gc\alpha$ for some $x = p(H_e^A)$, $x \in B_\delta$, we can find a $\tau \geqq \sigma_0$ such that $I_e^A(\tau) = gc\alpha$. On the other hand if $I_e^A(\tau) = gc\alpha$ for some $\tau \geqq \sigma_0$, then H_e^A receives attention at most once after τ. Define $M := \{x \in B_\delta | \exists \tau \geqq \sigma_0 (I_e(\tau) = gc\alpha$ for some H_e with $p'(\tau, H_e) = x\}$. M is again α-finite and we can map each $x \in B_\delta$ onto such a τ by an α-finite function g. Let ϱ be a bound on the range of g. It follows that $T_x - \varrho$ has order type $< gc\alpha$ for all $x \in B_\delta$. By the Sacks-Simpson-Lemma $\cup\{T_x - \varrho | x \in B_\delta\}$ must therefore be α-finite.

c) $gc\alpha < tp2\alpha < \alpha$, α satisfies (L) and $\delta = 2\gamma$. Take $\beta < \alpha$ such that $L_\beta \prec_{\Sigma_1} L_\alpha$, $p2\beta^{L_\beta} = \beta$, $\sigma_0 < \beta$, $p^{-1}[B_\delta] \subseteqq \beta$, $tp2\alpha < \beta$ and L_β contains all other parameters which are involved in the construction. In this case B_δ has the simple form $\{x | gc\alpha \cdot 2\gamma \leqq x < gc\alpha \cdot (2\gamma + 1)\}$. We show by induction on $\gamma < gc\alpha$, that we can find a $\tau < \beta$ for R_e with $p(R_e) = gc\alpha \cdot (2\gamma) + \nu$ such that R_e does not receive attention in β after τ. Assume this holds for all $\nu' < \nu$. Define a $\Sigma_2 L_\beta$ function $g: \nu \to \beta$ such that g maps each ν' on such a $\tau < \beta$. Rg g is bounded before β by some $\tau_1 < \beta$, because $cf2\beta^{L_\beta} = \beta$ (it is $gc\beta^{L_\beta} = gc\alpha$ because $L_\beta \prec_{\Sigma_1} L_\alpha$; if $cf2\beta^{L_\beta} < \beta$ then $p2\beta^{L_\beta} \leqq tp2\beta^{L_\beta} \leqq gc\alpha \cdot cf2\beta^{L_\beta} < \beta$, a contradiction).

For all stages τ the followers of R_e at τ are well ordered by the stages of their appointment. Here we call the ordinal which is represented by some follower p of R_e at τ in this well ordering the number of p.
We are finished, if R_e^A is satisfied before β. So assume that this is not the case. $\gamma:=\sup\{\delta+1 \mid \exists\tau\geqq\tau_1(\tau<\beta$ and R_e^A has a realized follower p at τ with number δ and p remains a follower of R_e^A in $[\tau,\beta))\}$. Further for each $\delta<\gamma$ we can find a $\tau'<\beta$ such that the follower p of R_e^A with number δ is always associated through the same $\langle\varrho,n\rangle$ with some Q_j in $[\tau',\beta)$. This is immediate for $\delta+1<\gamma$. Assume therefore that $\delta+1=\gamma$ and p is a follower of R_e^A with number δ in $[\tau,\beta)$. We look at the initial segment (with respect to $\prec$) S of $p2\alpha\times\omega$ which is covered by p in $[\tau,\beta)$. We map each $\langle\varrho,n\rangle\in S$ on the minimal $\sigma\in[\tau,\beta)$ such that p is associated at σ with some Q_j through some $\langle\varrho',n'\rangle$ with $\langle\varrho,n\rangle\prec\langle\varrho',n'\rangle$. Since $S\in L_\beta$, this can be done by a β-finite function which is bounded by some $\tau<\beta$. p is associated through some $\langle\varrho,n\rangle$ with some Q_j at τ and p doesn't leave this association in $[\tau,\beta)$ by the definition of S.
We observe further that $\gamma<\beta$, for if $\gamma=\beta$ we can map each $\delta<\gamma$ on the unique ϱ such that the follower of number δ is finally associated through some $\langle\varrho,n\rangle$ in β. We would get a one-one map $\beta\to p2\alpha$ which is $\Sigma_2 L_\beta$, contradicting $p2\beta^{L_\beta}=\beta$. Because $\gamma<\beta$ and $cf\,2\beta^{L_\beta}=\beta$, it follows that there is a $\tau_0<\beta$ such that R_e^A does not receive attention through some follower of number $<\gamma$ in $[\tau_0,\beta)$. By construction R_e^A has a follower p of order γ at the end of stage τ_0 which by definition of τ_0 is never cancelled in $[\tau_0,\beta)$. Further p is never realized in $[\tau_0,\beta)$ by definition of γ. Therefore p remains unrealized in $[\tau_0,\beta)$ and R_e^A does not receive attention in (τ_0,β).
We now have that for each $R_e\in B_\delta$ there exists a $\tau<\beta$ such that R_e does not receive attention in $[\tau,\beta)$ and therefore not in $[\tau,\alpha)$ because $L_\beta\prec_{\Sigma_1}L_\alpha$. Hence B_δ is discharged at β.

d) $gc\alpha<tp2\alpha<\alpha$, **$\alpha$ satisfies (L) and** $\delta=2\gamma+1$.
Choose β as before. If $H_e^A\in B_\delta$ receives attention at some $\tau\geqq\sigma_0$ with some $I_e^A(\tau)<gc\alpha$, then $\tau<\beta$ since $L_\beta\prec_{\Sigma_1}L_\alpha$. B_δ is therefore discharged at β.

The proof that each B_δ is discharged is now complete. It follows as in Lerman-Sacks that A, B are regular and not α-recursive.

Lemma 2. A, B are hyperregular.

Proof. Assume that A is not hyperregular. It follows that there is a regular α-cardinal κ and an e such that $f:=[e]^A\restriction\kappa$ is total on κ and unbounded. There is an unbounded set of indices e such that $[e]^A\restriction\kappa$ is the same function, so we may choose e such that $h_A(e)\geqq\kappa$. Choose σ_0 such that for $\tau\geqq\sigma_0$ $p'(\tau,H_e^A)=p(H_e^A)$ and no requirement of higher priority at τ receives attention at τ. Define an α-recursive $g:\kappa\to\alpha$ such that $[e]_\tau^{A^\tau}(\delta)\downarrow$ for $\tau=g(\delta)\geqq\sigma_0$. As a result of the presence of the requirement H_e^A and the definition of σ_0 these computations at stage $g(\delta)$ yield the final value $f(\delta)$. Therefore f is α-recursive and in particular bounded.

Finally one can prove as in Lemma 3.10 of Lerman-Sacks that each C weakly α-recursive in A and B is α-recursive.

Remark. In fact Lemma 3.10 of Lerman-Sacks seems to work without changes only if A, B are hyperregular. Otherwise for $c_C = [j_0]^A = [j_1]^B$ it is not clear, that we can find a stage $\tau \geqq \sigma_2$ for each x such that $L(\tau, j) = M(\tau, j) > x$. In this case M. Lerman suggests changing the proof as follows: One proves for $\beta < \alpha$ inductively that $c_C \cap \beta$ is α-recursive, using for the induction step the arguments of Lemma 3.10 and the fact that an α-recursive bounded set is an element of L_α so that an α-recursive initial segment of c_C can be computed by a single computation from A and B respectively (it is assumed that C is α-recursive, not only weakly α-recursive, in A and B). $M(\tau, j)$ has to be defined slightly differently for this proof.

§ 2. Nonzero Branching Degrees

Theorem 2. Let α be as in Theorem 1. Then there exist α-r.e. sets A, B, C such that C is not α-recursive, $C <_\alpha A + C$, $C <_\alpha B + C$ and $C = glb(A + C, B + C)$ (i.e. each D α-recursive in $A + C$ and $B + C$ is α-recursive in C). Further all these sets can be made hyperregular.

Remark. As we point out at the end of the proof, it seems to be necessary to make at least C hyperregular, even if one doesn't care to get hyperregular sets.

Proof of Theorem 2. We have the following requirements: $R_e^A \equiv [e]^{B+C} \neq A$, $R_e^B \equiv [e]^{A+C} \neq B$, $S_e \equiv \alpha - C \neq W_e$, $H_e^{A+C} \equiv ([e]^{A+C} \upharpoonright h_{A+C}(e)$ is bounded if it is total on $h_{A+C}(e))$ and H_e^{B+C} (analogous). For A, $B \subseteqq \alpha$ we define $A + B := \{x | (x = 2\gamma \wedge \gamma \in A) \vee (x = 2\gamma + 1 \wedge \gamma \in B)\}$. The priorities for requirements R_e, S_e are analogous to those for the previous R_e and the requirements H_e and functions h are handled in exactly the same way.
R_e^A is satisfied at σ, if at some $\tau < \sigma$ a follower p of R_e^A was put into A and the computation $[e]_\tau^{B^\tau + C^\tau}(p)$ was not destroyed in (τ, σ) or if at some $\tau < \sigma$ a follower p of R_e^A was realized and the computation $[e]_\tau^{B^\tau + C^\tau}$ was not destroyed in (τ, σ) and has value $\neq 1$.

As before one defines when R_e^A requires attention at σ, using $[e]_\sigma^{B^\sigma + C^\sigma}$ instead of $\{e\}_\sigma$.
S_e is satisfied at σ, if $C^\sigma \cap W_e^\sigma \neq \emptyset$.
S_e requires attention at σ, if S_e is not satisfied at σ, $e \leqq \sigma$, and

1) S_e is not persistant at σ or
2) S_e has a follower p and $p \in W_e^\sigma$ or
3) S_e has no follower at σ.

Requirements S_e have at most one follower p at the same time and one tries to make p a witness for $C \cap W_e \neq \emptyset$. $L(\sigma, j)$, $M(\sigma, j)$ are defined as before, writing $A + C$, $B + C$ instead of A, B.

Construction. Stage $\sigma > 0$: Take the requirement with highest priority at σ which requires attention at σ. Cancel all followers of requirements with lower priority at σ.
a) Requirements R_e, H_e are treated as before.
b) S_e. If S_e requires attention according to Case 1), cancel the follower of S_e. In Case 2) put p into C and cancel p as follower of S_e. In Case 3) appoint σ to be a follower of S_e.

The requirements S_e can be treated together with the R_e. In fact they are much more well behaved. If σ_0 is such that S_e is persistent after σ_0 and no requirement of higher priority receives attention after σ_0, then S_e receives attention at most twice after σ_0. It follows as before that A, B, C are regular, $A+C, B+C$ (and therefore A, B, C) are hyperregular and neither $A \leqq_{w\alpha} B+C$ nor $B \leqq_{w\alpha} A+C$ holds. Further C is not α-recursive due to the requirements S_e. It remains only to show:

Lemma 3. Assume $f = c_D = [j_0]^{A+C} = [j_1]^{B+C}$ is total. Then $f \leqq_{w\alpha} C$.

Proof. Go to a stage σ_0 such that Q_j is persistent after σ_2 and no requirement of priority $\leqq p_1(j)$ receives attention after σ_0. We show that for all x, y $f(x) = y \Leftrightarrow \langle L_\alpha, C\rangle \models \Psi(x,y)$ with the following Σ_1-formula Ψ: $\Psi(x,y) \equiv \exists \tau \varrho L K H (\sigma_0 \leqq \tau \leqq \varrho \wedge L(\tau,j) = M(\tau,j) > x \wedge L = \{p \mid p \text{ is follower of some requirement } S_e \text{ at } \tau\} \wedge L = K \cup H \wedge K \subseteqq C \wedge H \subseteqq \alpha - C \wedge K \subseteqq C^\varrho \wedge$ (no computation $[j_0]_\tau^{A^\tau+C^\tau}(z)$, $[j_1]_\tau^{B^\tau+C^\tau}(z)$ with $z < L(\tau,j)$ is destroyed in $[\tau,\varrho]) \wedge [j_0]_\tau^{A^\tau+C^\tau}(x) \cong y)$.
We observe first that for all x there exists an y such that $\langle L_\alpha, C\rangle \models \Psi(x,y)$: Because $A+C, B+C$ are hyperregular there exists τ' such that $[j_0]_{\tau'}^{A+C}(z)$, $[j_1]_{\tau'}^{B+C}(z)$ converge for all $z \leqq x+1$. Take $\tau \geqq \tau'$ such that $A^\tau + C^\tau \cap \tau' = A + C \cap \tau'$ and $B^\tau + C^\tau \cap \tau' = B + C \cap \tau'$. Then the computations $[j_0]_\tau^{A^\tau+C^\tau}(z)$, $[j_1]_\tau^{B^\tau+C^\tau}(z)$ with $z \leqq x+1$ give the final value and are never destroyed, so we just have to wait for a stage ϱ where $L \cap C \subseteqq C^\varrho$.
To prove that $\langle L_\alpha, C\rangle \models \Psi(x,y) \Rightarrow f(x) = y$ we proceed as in Lerman-Sacks. One shows that the computation on one side of the equality for argument x is only destroyed for the n-th time by a follower of some R_e at stage $\tau_n > \varrho$ if the computation on the other side is already reestablished. Then we have to eliminate the possibility that this argument collapses because at some stage $> \tau_n$ a follower of some S_e, coming into C, destroys the computation on either side of the equality which was reestablished at τ_n (this is possible because C occurs on both sides). But this can't happen, because one can prove inductively that all followers of any requirement R_e or S_e at some τ_n are in fact $< \tau$. Now a follower $< \tau$ of some S_e, if it ever comes into C, comes into C before ϱ, which is less than τ_n.

Remark. It is tempting to use Lerman's trick from the end of §1 to avoid the assumption that the sets are hyperregular. But in this case it is not possible to prove by induction on $\beta < \alpha$ that $c_D \cap \beta$ is α-recursive in C and therefore an element of L_α. In fact for every α-r.e. non-complete set C which is not hyperregular, there exists a bounded set M which is α-recursive in C (not only $\Delta_1 \langle L_\alpha, C\rangle$) and not an element of L_α (see [8]).

§ 3. Minimal Pairs of Functionals

We use here the notation of Harrington [4], Sacks [11]. Let F^{n+2} be a given normal functional where $n \geqq 1$. We want to construct functionals G^{n+2}, H^{n+2} r.e. in F and some subindividual r^{n-1} which form a minimal pair.
In order to translate a solution to Post's Problem from α-recursion theory into recursion in higher types, Harrington [4] introduced the following admissible structure $\mathfrak{A}$: Let $\alpha := \lambda^F_{n-1}$ be the order type of the set S of ordinals subconstructive in F and $\tau : \alpha \to S$ the isomorphism. Define $T \subsetneq \omega \times \alpha \times \alpha$ by $T = \{\langle e, \beta, \sigma\rangle \mid e$ is Godel number of a Σ_1 formula $\Phi(x)$ and $M_{\tau\sigma}(F) \models \Phi(\tau\beta)\}$ where $M(F)$ is the constructible hierarchy relativized to F. $\mathfrak{A}$ is then defined to be the admissible structure $\langle L_\alpha(T), \varepsilon, T\rangle$. For any $A \subsetneq \alpha$ $(\tau A)^H$ is the type $(n+2)$ object $\{H^H_\sigma \mid \sigma = \tau\gamma$ for some $\gamma \in A\}$ (H^F is the Harrington hierarchy). Harrington proved that for each $C \subsetneq \alpha$, which has the properties that C is regular and hyperregular over $\mathfrak{A}$ and τC subgeneric over F, the following holds: $D \subsetneq \alpha$ is $\Delta_1\langle \mathfrak{A}, C\rangle \Leftrightarrow (\tau D)^H \leqq \langle F, (\tau C)^H, H^F_\sigma\rangle$ for some σ subconstructive in F (we write $\leqq$ for "Kleene-recursive in").
We assume for the following that there is a well ordering $\prec$ of the subindividuals $SI := tp(n-1)$ which is recursive in F. So in ZFC we restrict our attention to functionals F which are "strong enough" whereas with $V = L$ we can already find a well ordering $\prec$ which is recursive in each normal F.
With the help of $\prec$ we can code the subindividuals by subconstructive ordinals: Define

$$\{z'\}(E^{n+1}, \prec, x^0, r^{n-1}, s^{n-1}) \simeq \begin{cases} \{x\}(E, \prec, s) & \text{if } s \prec r \text{ and } x \text{ is a Kleeneindex of the appropriate form} \\ 0 & \text{otherwise.} \end{cases}$$

The recursion theorem gives us a z such that $\{z\}(E^{n+1}, \prec, r^{n-1}) \simeq E^{n+1}(\lambda s^{n-1}\{z'\}(E, \prec, z, r, s))$. One shows by induction over $\prec$ that for all r $\{z\}(E, \prec, r)\downarrow$ and $\bigcirc(r) := |\{z\}(E, \prec, r)|^{E,\prec}$ (which is the length of this computation) is the desired code for r. The function $\bigcirc: SI \to On$ is strictly increasing and continuous. $\mathrm{Rg}\bigcirc$ is bounded by

$$\varrho_0 := |E^{n+1}(\lambda r^{n-1}\{z\}(E, \prec, r))|^{E,\prec} \text{ which is} < \kappa^F_0.$$

We can define Σ_1 formulae Φ_1, Φ_2 such that $\bigcirc(r) = \delta \Leftrightarrow M_{\kappa^F_{n-1}}(F) \models \Phi_1(\delta, r)$ for $r \in SI$, $\delta < \kappa^F_{n-1}$ (obvious) and such that for $\delta < \kappa^F_{n-1}$

$$(\delta \notin \mathrm{Rg}\bigcirc \Leftrightarrow M_{\kappa^F_{n-1}}(F) \models \Phi_2(\delta)).$$

We take $\Phi_2(\delta) :\equiv \delta \geqq \varrho_0 \vee (\delta < \varrho_0 \wedge \exists r^{n-1} s^{n-1} \beta\gamma$

$$(\Phi_1(\beta, r) \wedge \Phi_1(\gamma, s) \wedge \beta < \delta < \gamma \wedge \neg\exists t^{n-1}(r \prec t \wedge t \prec s)))$$

where $\neg\exists t^{n-1}(r \prec t \wedge t \prec s)$ is recursive in F as a predicate of r, s and therefore expressible by a Σ_1 formula in $M_{\kappa^F_{n-1}}(F)$.
It follows that for each r^{n-1} we can find a subconstructive σ such that $r \leqq \langle F, H^F_\sigma\rangle$: Take $\sigma := \bigcirc(r)$, then

$$r(x^{n-2}) = m \Leftrightarrow M_{\kappa_0\langle F, H^F_\sigma\rangle}\langle F, H^F_\sigma\rangle \models \exists s^{n-1}\delta z(\delta = \bigcirc(s) \wedge z = H^F_\delta \wedge z = H^F_\sigma \wedge s(x) = m)$$

where the latter can be written as a Σ_1 formula with parameters F, H_σ^F. It follows by Harrington [4] that the graph of r is r.e. in F, H_σ so that r itself is recursive in F, H_σ (by Gandy-selection).

The existence of $\prec \leqq F$ implies further that $p1\mathfrak{A}$ (the Σ_1 projection of α in $\mathfrak{A}$) is less than α: Let $Q \subseteq S \times S$ be defined by $Q(\delta,\gamma) \leftrightarrow \exists r^{n-1}(\delta = |r|^F \wedge \gamma = \mathrm{O}(r))$. Then Q is obviously Σ_1 over $M_{\kappa_{n-1}^F}(F)$ and $\forall\delta \in S \exists\gamma \in S\, Q(\delta,\gamma)$. So $\tau^{-1}[Q]$ is $\Sigma_1\mathfrak{A}$ and the $\Sigma_1\mathfrak{A}$ uniformization of this yields a projection of α into $\tau^{-1}(\varrho_0)$ which is less than α. Since $p1\mathfrak{A} < \alpha$ we can be sure that $\mathfrak{A}$ is a nonrefractory structure so that Theorem 1 gives us a minimal pair A, B in $\mathfrak{A}$ of $\mathfrak{A}$-r.e., hyperregular sets which has then the property that $C \subseteq \alpha$, $C\Delta_1\langle\mathfrak{A}, A\rangle$, $C\Delta_1\langle\mathfrak{A}, B\rangle \Rightarrow C\Delta_1\mathfrak{A}$. Further one can easily insure that $(\tau A)^H$, $(\tau B)^H$ are subgeneric in F by adding requirements to the construction in $\mathfrak{A}$ which are treated in a similar manner to the requirements H_e (see [4]). Since $C\Sigma_1\mathfrak{A}$ implies that $(\tau C)^H$ is r.e. in $\langle F, r\rangle$ for some r^{n-1} we may take $G:=(\tau A)^H, H:=(\tau B)^H$ and get:

Theorem 3. Assume that a well ordering $\prec$ of $tp(n-1)$ is recursive in the normal object F^{n+2}. Then there exist functionals G^{n+2}, H^{n+2} both r.e. in $\langle F, r\rangle$ for some r^{n-1} but not recursive in $\langle F, s\rangle$ for any s^{n-1} such that: Any $L^{n+2} \subseteq \{H_\sigma^F | \sigma$ subconstructive$\}$ where $L \leqq \langle F, G, r_0\rangle$ and $L \leqq \langle F, H, r_1\rangle$ for some r_0^{n-1}, r_1^{n-1} is *recursive in* $\langle F, s\rangle$ *for some* s^{n-1}.

Using a weaker reducibility we get rid of the restriction $L^{n+2} \subseteq \{H_\sigma^F | \sigma$ subconstructive$\}$. We define relative to the fixed functional F:

$$G_1^{n+2} \leqq_{ws} G_2^{n+2} \Leftrightarrow \exists e^0 s^{n-1} \forall j^0 r^{n-1} (G_1(\lambda y^n \{j\}(F, G_2, y, r)) \simeq \{e\}(G_2, F, s, j, r)),$$

which is a restriction of "weakly Kleene-recursive in" to subindividuals (assuming stronger axioms than ZFC Normann proved in [9] the existence of a minimal pair for "weakly Kleene-recursive in").
Writing $\{j\}_p$ for a standard indexing of primitive recursive functionals we define for any G^{n+2}.
$M_G := \{\langle\delta, \gamma, i\rangle | \delta, \gamma \in S \wedge i \in \{0, 1\} \wedge \exists jmr(\delta = |\langle mr\rangle|^F \wedge \gamma = \mathrm{O}(\langle j, m, r\rangle) \wedge f^{n+1} := \lambda y^n \mu x^0(\{j\}_p(H_\delta^F, r, y, x) = 0)$ is total $\wedge\, G(f^{n+1}) = i\}$ and

$M'_G := \{\langle\gamma, \delta, i\rangle | \langle\tau\gamma, \tau\delta, i\rangle \in M_G\}$.

Lemma 4. Assume that $A \subseteq \lambda_{n-1}^F$ is $\Sigma_1\mathfrak{A}$, regular, hyperregular and $(\tau A)^H$ is subgeneric over F. Then, for any G^{n+2}, the following holds

a) $G \leqq_{ws} (\tau A)^H \Rightarrow M'_G$ is $\Delta_1\langle\mathfrak{A}, A\rangle$,

b) M'_G is $\Delta_1\mathfrak{A} \Rightarrow G \leqq_{ws} F$.

Proof. a) M_G is $\Delta_1 M_{\kappa_{n-1}^F}\langle F, (\tau A)^H\rangle$. To show it "$\Sigma_1$" we use the fact that $(\tau A)^H$ subgeneric over F implies (because of $\prec$) $\kappa_{n-1}^{\langle F, (\tau A)^H\rangle} \leqq \kappa_{n-1}^F$. For "$\Pi_1$" we use the fact that "$\delta \notin \mathrm{Rg}$" is $\Sigma_1 M_{\kappa_{n-1}^F}(F)$ (it was for this part of the proof that we coded the δ in M_G). As in Harrington [4] it follows that M'_G is $\Delta_1 \langle \mathfrak{A}, A \rangle$.

b) One can show easily that $K := \{\langle j^0, r^{n-1}, i^0 \rangle | G(\lambda y^n \{j\}(F, y, r)) \simeq i\}$ is $\Sigma_1 M_{\kappa_{n-1}^F}(F)$ and therefore r.e. in F together with some s^{n-1}. The result follows by Gandy-selection.

Remark. It is obvious that for any G^{n+2}, H^{n+2}, s^{n-1} $G \leqq \langle F, H, s \rangle \Rightarrow G \leqq_{ws} H$. Because "$r \in O^F \wedge f^{n+1} = H^F_{|r|}$" is as predicate of f, r recursive in F one gets further that $G \leqq_{ws} H \Rightarrow G \leqq \langle F, H, s \rangle$ for some s^{n-1} if G is of the special form $G = (\tau A)^H$, $A \subseteq \alpha$.
From Theorem 1 and Lemma 4 we get

Theorem 4. Assume that a well ordering $\prec$ of $tp(n-1)$ is recursive in the normal object F^{n+2}. Then there exist G^{n+2}, H^{n+2} both r.e. in $\langle F, r \rangle$ for some r^{n-1} but not recursive in $\langle F, s \rangle$ for any s^{n-1} such that for all functionals L^{n+2}: $L \leqq_{ws} G \wedge L \leqq_{ws} H \Rightarrow L \leqq_{ws} F$.
The analogous reasoning applied to Theorem 2 gives

Theorem 5. Assume that a well ordering $\prec$ of $tp(n-1)$ is recursive in the normal object F^{n+2}. Then there exists G^{n+2}, H^{n+2}, K^{n+2} r.e. in F and some r^{n-1} which are not recursive in $\langle F, s \rangle$ for any s^{n-1} such that: $K <_{ws} G$, $K <_{ws} H$ and for all L_{n+2}:

$$L \leqq_{ws} G \wedge L \leqq_{ws} H \Rightarrow L \leqq_{ws} K.$$

§4. Minimal α-degrees for some Σ_1-admissible α

We assume for this chapter that the reader is acquainted with Shore [12], so we merely have to describe the changes which make Shore's construction work for a larger class of α (α always admissible).
We are going to use in this and the following chapter some simple facts of weakly inadmissible recursion theory, see [3], [8]: Let β be any limit ordinal and $A \subseteq \beta$ be regular over L_β, $\mathfrak{B} := \langle L_\beta, A \rangle$. Assume $\kappa := \Sigma_1 cf \mathfrak{B} \geqq \mathfrak{B}^* := \Sigma_1$ projection of $\mathfrak{B}$. Then there is a one-one onto projection $P: \beta \to \kappa$ which is $\Delta_1 \mathfrak{B}$. We can further define a predicate $T \subseteq \kappa$, $T \Delta_1 \mathfrak{B}$, T regular over L_κ such that $\mathfrak{A} := \langle L_\kappa, T \rangle$, the "admissible collaps of $\mathfrak{B}$", is admissible and for any $B \subseteq \kappa: B \Sigma_1(\Delta_1) \mathfrak{A} \Leftrightarrow B \Sigma_1(\Delta_1) \mathfrak{B}$.
Assume now that α is Σ_1-admissible and $p2\alpha \leqq cf 2\alpha$. Let $A \subseteq \alpha$ be a regular complete α-r.e. set and define $\mathfrak{B} := \langle L_\alpha, A \rangle$. When $cf 2\alpha < \alpha$ we have that $\mathfrak{B}$ is inadmissible, so we have to overcome the following difficulties: First if the construction proceeds by recursion in $\mathfrak{B}$ in α many steps, the construction need not be $\mathfrak{B}$-recursive. Second the Sacks-Simpson Lemma might fail in $\mathfrak{B}$ (see [3, 8]).
For any partial $f: \alpha \to \alpha$ we have $f \Sigma_2 L_\alpha \Leftrightarrow f \Sigma_1 \mathfrak{B}$, so $\mathfrak{B}$ is weakly inadmissible which means that $\kappa := cf 2\alpha = \Sigma_1 cf \mathfrak{B} \geqq \mathfrak{B}^* = p2\alpha =: \gamma$.

Let $\mathfrak{A} = \langle L_\kappa, T\rangle$ be the admissible collaps of $\mathfrak{B}$ (in the special case where α is Σ_2-admissible, $\mathfrak{A}$ is essentially the same as $\mathfrak{B}$). Take $g:\kappa\to\alpha$ to be cofinal, increasing, continuous and $\Sigma_1\mathfrak{B}$. For $\delta_0 := tp2\mathfrak{A}$ there is a $\Sigma_1\mathfrak{A}$ function $h':\kappa\times\delta_0\to\kappa$ such that $\forall\beta<\delta_0\exists\sigma_0<\kappa\forall x\leqq\beta\forall\sigma\geqq\sigma_0(h'(\sigma,x)=h'(\sigma_0,x))$ and $h=\lim_{\sigma\to\kappa} h'(\sigma,\cdot)$ maps δ_0 onto κ. We define $k:\kappa\times\delta_0\to\alpha$ by $k(\sigma,\varepsilon)=P^{-1}(h'(\sigma,\varepsilon))$. Then k is $\Sigma_1\mathfrak{B}$ and a tame approximation for the function $k_0=\lim_{\sigma\to\kappa} k(\sigma,\cdot)$ where k_0 maps δ_0 onto α. A slight twist as in Shore [12] assures that for each $\varepsilon<\delta_0$ $k(\cdot,\varepsilon)$ settles down at its final value at some $\sigma<\kappa$ which is not a limit ordinal.

The construction takes place in $\mathfrak{B}$ in κ many steps and the trees are the same as in [12]. Since $\kappa=\Sigma_1 cf\mathfrak{B}$ we can define a function $H:\kappa\to L_\alpha$ in $\mathfrak{B}$ by the scheme $H(\sigma) = G(H\restriction\sigma,\sigma)$ with $G\Sigma_1\mathfrak{B}$ such that H is $\Sigma_1\mathfrak{B}$ and each initial segment of H is α-finite (which carries us through the limit cases of the construction as in [12]). Then we have to assure that $B=\bigcup_{\sigma<\kappa}\beta_\sigma$ is regular, which we need for Lemma 6 below. For this purpose we make $\lg\beta_\sigma\geqq g(\sigma)$ for all $\sigma<\kappa$, which causes some additional trouble in the construction.

Construction. *Stage* 0: Set $f_0=0$, $T_0^0=\mathrm{id}$, $\beta_0=\phi$.

Stage $\sigma=\gamma+1$: Let η be the least $\varepsilon<f\gamma$ with $k(\sigma,\varepsilon)\neq k(\gamma,\varepsilon)$ ($\eta=f\gamma$ if such a $\varepsilon<f\gamma$ doesn't exist).

a) If $\lg\beta_\gamma\geqq g(\sigma)$ we take as in Shore's construction the least $i<\eta+1$ such that β_γ has no proper extension on T_i^γ.

b) If $\lg\beta_\gamma<g(\sigma)$ and $\beta_\gamma=T_\eta^\gamma(\tau)$ has extensions α_0, α_1 of $T_\eta^\gamma(\tau*0)$ and $T_\eta^\gamma(\tau*1)$ respectively on T_η^γ which have length$>g(\sigma)$ we take $f\sigma=\eta+1$.

c) Otherwise $f\sigma$ is some $i\leqq\eta$ (see the remark following the construction) such that $i=j+1$ and a sequence β with $\beta_\gamma\subsetneqq\beta$ and $\lg\beta\leqq g(\sigma)$ exists on T_i^γ such that β has no extension on T_i^γ but for τ where $T_j^\gamma(\tau)=\beta$, it is the case that $T_j^\gamma(\tau*0)$ and $T_j^\gamma(\tau*1)$ are defined and have extensions α_0, α_1 of length$>g(\sigma)$ on T_j^γ.

Case 1: If $f\sigma=\eta+1$ we have by definition of $f\sigma$ incompatible sequences α_0, α_1 of length$>g(\sigma)$ on T_η^γ which extend β_γ. One of them, say α_0, is incompatible with $R_{k(\sigma,\eta)}$. Take $\beta_\sigma=\alpha_0$, $T_i^\sigma=T_i^\gamma$ for $i<f\sigma$ and $T_{f\sigma}^\sigma=\mathrm{Sp}(T_\eta^\sigma, k(\sigma,\eta), \beta_\sigma)$.

Case 2: $f\sigma\leqq\eta$. If $\lg\beta_\gamma\geqq g(\sigma)$ we have $f\sigma=j+1$ and $T_{j+1}^\gamma=\mathrm{Sp}(T_j^\gamma, k(\gamma,j), \beta)$ for some $\beta\subsetneqq\beta_\gamma$. Further β_γ has extensions $\alpha_0=\beta_\gamma^0$, $\alpha_1=\beta_\gamma^1$ on T_j^γ. We take one of these, call it α_n, as determined by 1.8 of Shore [12].

If $\lg\beta_\gamma<g(\sigma)$ the definition of $f\sigma=j+1$ in c) implies that $T_{f\sigma}^\gamma=\mathrm{Sp}(T_j^\gamma, k(\gamma,j), \tilde{\beta})$ for some $\tilde{\beta}\subsetneqq\beta_\gamma$ and we have $\beta=T_j^\gamma(\tau)$ on $T_{f\sigma}^\gamma$, $\beta\supsetneqq\beta_\gamma$, with extensions $T_j^\gamma(\tau*0)$, $T_j^\gamma(\tau*1)$ on T_j^γ. We take again one of these, following 1.8 of Shore [12], and then the corresponding extensions $\alpha_n\supsetneqq T_j^\gamma(\tau*n)$ on T_j^γ of length$>g(\sigma)$ which exist by the definition of $f\sigma$. We define

$$\beta_\sigma=\alpha_n,\ T_i^\sigma=T_i^\gamma \text{ for } i<f\sigma \text{ and } T_{f\sigma}^\sigma=Fu(T_j^\gamma,\alpha_n).$$

Stage σ, σ limit: exactly as in Shore's paper.

Remark. For c) where we defined $f\sigma$ as "some $i \leqq \eta$ such that..." we have to show that such an $i \leqq \eta$ exists and to explain how this choice is made unique. To prove the existence let i_0 be the minimal $i \leqq \eta$ such that a sequence β exists on T_i^γ with $\beta_\gamma \subsetneqq \beta$, $\lg\beta \leqq g(\sigma)$ and β has no proper extension on T_i^γ. We can find such an i because $i = \eta$ already has this property:
Assume this isn't true. For some τ we have $\beta_\gamma = T_\eta^\gamma(\tau)$ and so $\alpha'_0 = T_\eta^\gamma(\tau * 0)$, $\alpha'_1 = T_\eta^\gamma(\tau * 1)$ are defined because we are in the case $\lg\beta_\gamma < g(\sigma)$. Take for $n = 0, 1$ $\delta_n \leqq g(\sigma) + 1$ minimal such that $T_\eta^\gamma(\tau * n * \bar{O}_{\delta_n})\uparrow$ where $\bar{O}_\delta$ is a sequence of O's of length δ. If this $\delta_n \leqq g(\sigma) + 1$ doesn't exist we get an extension $\alpha_n := T_\eta^\gamma(\tau * n * \bar{O}_{g(\sigma)+1})$ of α'_n which has length $> g(\sigma)$. If $\delta_n \leqq g(\sigma) + 1$ exists it is a successor, say $\delta'_n + 1$, and $\alpha_n := T_\eta^\gamma(\tau * n * \bar{O}_{\delta'_n})$ has length $> g(\sigma)$ (otherwise $\beta = \alpha_n$, which would contradict the above assumption). So in any case we get extensions α_n of $T_\eta^\gamma(\tau * n)$ on T_η^γ for $n = 0,1$ which have length $> g(\sigma)$. But then $f\sigma$ was already defined by b), which yields a contradiction.
We have now shown that the minimal $i_0 \leqq \eta$, which we defined above, actually exists. Because of its minimality i_0 has the form $j + 1$. Further by definition of i_0 there is a sequence β on T_{i_0} with $\beta_\gamma \subsetneqq \beta$, $\lg\beta \leqq g(\sigma)$ and β has no proper extension on $T_{i_0}^\gamma$. We have then $\beta = T_j^\gamma(\tau)$ for some τ and arguing as before for T_η^γ (using the minimality of i_0) we get extensions α_n of $T_j^\gamma(\tau * n)$ lying on T_j^γ with length $> g(\sigma)$. Therefore i_0 is a witness that an $i \leqq \eta$ with the properties demanded in c) exists.

The reason why we didn't define $f\sigma = i_0$ in Case c) is that we needed the quantifier $\forall\bar{\alpha}(\lg\bar{\alpha} \leqq g(\sigma) \rightarrow \ldots$ in order to express the fact that i_0 was minimal. But these $\bar{\alpha}$ might be spread over all stages of $\langle L_\alpha, A\rangle$, so the quantification could lead us away from the blessed path of $\mathfrak{B}$-recursiveness. On the other hand we can define a predicate P which is $\Sigma_1\mathfrak{B}$, such that $P(F \restriction \sigma, i)$ says that i has the properties which are demanded in Case c), where $F: \kappa \rightarrow L_\alpha$ is the $\Sigma_1\mathfrak{B}$ function which describes the construction. So we may leave the choice of i to a $\Sigma_1\mathfrak{B}$ uniformization of the predicate P (see Devlin [2]). The choice of the sequences β, α_0, α_1 is treated analogously.

Since the whole construction is $\mathfrak{B}$-recursive, the regular set $B := \bigcup_{\sigma<\kappa} \beta_\sigma$ is recursive in A.

Lemma 5. $I_\varepsilon := \{\sigma \mid f\sigma < \delta_0\}$ is bounded in κ for all $\varepsilon < \delta_0$.

Proof. The function $f: \kappa \rightarrow \kappa$, $\sigma \rightarrow f\sigma$ is $\Sigma_1\mathfrak{B}$, therefore $\Sigma_1\mathfrak{A}$. So we can work for this proof in the admissible structure $\mathfrak{A}$. Since we use the tame Σ_2 projection, we don't need skolem hull arguments. If $tp2\mathfrak{A} = \delta_0 \leqq gc\mathfrak{A}$ and $\sigma_0 < \kappa$ is such that for a given $\varepsilon < \delta_0$ $h(\sigma, \delta)$ always gives the final value after σ_0 for all $\delta \leqq \varepsilon$, one proves by induction on δ that $I_\delta - \sigma_0$ has order type $< \delta^+$ (δ^+ is the next $\mathfrak{A}$-cardinal after δ). For $\delta = \delta' + 1$ we use the fact that if $I_{\delta'} - \sigma_0$ has order type ϱ then $I_\delta - \sigma_0$ has an order type $\leqq 3\varrho + 3$ (see Shore [12]). For limit stages one applies the Sacks-Simpson-Lemma. If $tp2\mathfrak{A} > gc\mathfrak{A}$ and $cf2\mathfrak{A} = \kappa$, we prove in an analogous way that $I_\delta - \sigma_0$ has order

type $<\kappa$ for all $\delta \leqq \varepsilon$. The argument is the same for successor and limit stages, using the fact that $cf\,2\mathfrak{A} = \kappa$.
The remaining case is $cf\,2\mathfrak{A} < \kappa$, $tp\,2\mathfrak{A} > gc\mathfrak{A}$ which implies that $tp\,2\mathfrak{A} = \delta_0 = gc\mathfrak{A} \cdot cf\,2\mathfrak{A}$ (see § 1). We proceed again by induction. If $\sigma_0 < \kappa$ is a bound for $I_{gc\mathscr{A}\cdot\varrho}$ and h gives the correct values for $\delta < gc\mathfrak{A}\cdot(\varrho+1)$ after σ_0, the standard argument shows that $I_{gc\mathscr{A}\cdot\varrho+\gamma} - \sigma_0$ has order type $<\gamma^+$ for all $\gamma < gc\mathfrak{A}$. Define $M \subseteqq gc\mathfrak{A} \times gc\mathfrak{A}$ by $\langle x, y\rangle \in M \leftrightarrow \exists\sigma(\sigma > \sigma_0 \wedge (f\sigma < gc\mathfrak{A}\cdot\varrho + \gamma$ for the x-th time after $\sigma_0))$. M is $\mathfrak{A}$-r.e. and bounded. Therefore $M \in L_\kappa$ (we have $\mathfrak{A}^* = \kappa$ in this case) and we may map each $\langle x, y\rangle \in M$ onto the σ from the definition of M by a map $g \in L_\kappa$. Because Rgg is bounded below κ, $I_{gc\mathscr{A}\cdot(\varrho+1)}$ is bounded below κ. Finally for the case $I_{gc\mathscr{A}\cdot\lambda}$, $\lambda < cf\,2\mathfrak{A}$ limit, we merely use the definition of $cf\,2\mathfrak{A}$.
It follows as in [12], that for every $\varepsilon < \delta_0$ there is a last stage σ_ε for which $f\sigma_\varepsilon = \varepsilon$ and that $\sigma_\varepsilon < \sigma_{\varepsilon'}$ for $\varepsilon < \varepsilon'$. Further his proof that B is not α-recursive works here as well.

Lemma 6. If $[e]^B$ is a representing function than it is either α-recursive or B is α-recursive in it.

Proof. Take η such that $\lim\limits_{\sigma\to\kappa} k(\sigma,\eta) = e$. Let σ_0 be the last stage where $f\sigma_0 = \eta + 1$. Then we have $T_{\eta+1} = \lim T^\sigma_{\eta+1} = T^{\sigma_0}_{\eta+1}$ and $T_\eta = \lim T^\sigma_\eta = T^{\sigma_0}_\eta$.
a) If $T_{\eta+1} = \mathrm{Fu}(T_\eta, \beta_{\sigma_0})$, $[e]^B$ is α-recursive.
b) $T_{\eta+1} = \mathrm{Sp}(T_\eta, e, \beta_{\sigma_0})$.
For this case we take a closer look at the definition of a splitting tree. Define $\tilde{g}$ to be partial α-recursive such that $\tilde{g}(e_0, e, \tau)$ is defined iff there exists an tupel $\langle \tau_0, \tau_1, \alpha_0, \alpha_1, x, y_0, y_1\rangle$ with $\tau * j \subseteqq \tau_j$, $\{e_0\}(\tau_j) = \alpha_j$ and $[e]^{\alpha_j}(x) = y_j$ for $j = 0, 1$ where $y_0 \neq y_1$. If such a tupel exists, $\tilde{g}(e_0, e, \tau)$ gets one of these as value (by a $\Sigma_1 L_\alpha$ uniformization). $\mathrm{Sp}(T_\eta, e, \beta_{\sigma_0})$ is then defined for the tree $T_\eta = \{e_0\}$ by the recursion theorem, using the function $\tilde{g}$ for the definition at sequences $\tau * 0$, $\tau * 1$.
For τ, β *where* $T_{\eta+1}(\tau) = \beta$, $\lg\tau = \delta$ we define (with the help of $\tilde{g}$) an α-recursive function $h_\beta: \delta \to L_\alpha$ such that

$$h_\beta(\varrho) = \langle x, y\rangle \Leftrightarrow (\tilde{g}(e_0, e, \tau \upharpoonright \varrho) = \langle \tau_0, \tau_1, \alpha_0, \alpha_1, x, y_0, y_1\rangle \wedge ((y = y_0 \wedge \alpha_0 \subseteqq \beta) \vee (y = y_1 \wedge \alpha_1 \subseteqq \beta))).$$

Since h_β is total on δ, h_β is in fact α-finite. So $\mathrm{Rg}h_\beta$ is α-finite as well and for any sequence β on $T_{\eta+1}$ we have that β is an initial segment of c_B iff $\mathrm{Rg}h_\beta \subseteqq [e]^B$. Therefore $K \subseteqq c_B \leftrightarrow \exists\tau\beta(T_{\eta+1}(\tau) = \beta \wedge K \subseteqq \beta \wedge \mathrm{Rg}h_\beta \subseteqq [e]^B)$ and B is α-recursive in $[e]^B$.

We have now proved

Theorem 6[1]**.** Assume that α is admissible and $p2\alpha \leqq cf\,2\alpha$. Then a non zero minimal degree exists which is recursive in O'.

[1] Work on this result has been done independently by R. A. Shore.

Remark. A function $f:\delta\to\alpha$ is called a S_3 projection if f is onto and there exists a Σ_2 function $f':\alpha\times\delta\to\alpha$ such that $\forall x<\delta(f(x)=\lim_{\sigma\to\alpha} f'(\sigma,x))$ (see Lerman [7]). $s3p(\alpha)$ is the least δ such that there exists a S_3 projection $f:\delta\to\alpha$. Assume now that $s3p(\alpha)\leqq cf2\alpha$ and $f:s3p(\alpha)\to\alpha$ is a S_3 projection with the approximation $f':\alpha\times s3p(\alpha)\to\alpha$. Define $h:cf2\alpha\times s3p(\alpha)$ by $h(\sigma,x)=f'(g(\sigma),x)$ where $g:cf2\alpha\to\alpha$ is cofinal and $\Sigma_1\mathfrak{B}$ as before. Then h is $\Sigma_1\mathfrak{B}$ because f' is $\Sigma_1\mathfrak{B}$ and we further have $\forall y\in\alpha\ \exists\sigma\exists x\ (h(\sigma,x)=y)$. The $\Sigma_1\mathfrak{B}$ uniformization of h^{-1} yields a $\mathfrak{B}$-recursive $1-1$ map $g:\alpha\to cf2\alpha$ which shows that $p2\alpha\leqq cf2\alpha$. Therefore the theorem includes the case $s3p(\alpha)\leqq cf2\alpha$, in particular the case $s3p(\alpha)=\omega$ where maximal sets exist ([7]).

§5. Generalization to some Weakly Inadmissible Structures

Let $\beta=\omega\cdot\gamma$ be a limit ordinal and $C\subseteqq\beta$ such that C is regular over J_γ (see Devlin [2] for the definition of the Jensen hierarchy J). A recursion theory on $\mathfrak{B}:=\langle J_\gamma,C\rangle$ is then defined by taking $\Sigma_1\mathfrak{B}$ subsets of β as the $\mathfrak{B}$-r.e. sets (see Sy Friedman [3]). The reducibilities $\leqq_{w\mathscr{B}},\leqq_{\mathscr{B}}$ are defined as in α-recursion theory. We consider here the case where $\mathfrak{B}$ is weakly inadmissible, which means $\beta>\kappa:=\Sigma_1\,cf\,\mathfrak{B}\geqq\mathfrak{B}^*=:\gamma$ (see [8] for examples). Define the admissible collaps $\mathfrak{A}=\langle L_\kappa,T\rangle$ as in §4.
We call a set $A\subseteqq\kappa$ β-immune if $\forall K\in J_\gamma\ ((K\subseteqq A\vee K\subseteqq\kappa-A)\to K\in L_\kappa)$. The advantage of β-immune sets A is, that for every $C\subseteqq\kappa: C\leqq_{w\mathfrak{B}}A\Rightarrow C\leqq_{w\mathfrak{A}}A$ and $A\leqq_{\mathscr{A}}C\Rightarrow A\leqq_{\mathscr{B}}C$. The following holds: For every $A\subseteqq\kappa$ we get uniformly a β-immune set $\tilde{A}\subseteqq\kappa$ such that $A=_{\mathscr{A}}\tilde{A}$ and $A\Sigma_1(\Delta_1)\mathfrak{A}\Leftrightarrow\tilde{A}\Sigma_1(\Delta_1)\mathfrak{A}$ (see [8]).

Theorem 7. Let $\mathfrak{B}=\langle J_\gamma,C\rangle$ be weakly inadmissible and assume that the admissible collaps $\mathfrak{A}$ of $\mathfrak{B}$ is not refractory (for example assume that $\Sigma_1\,cf\,\mathfrak{B}>\mathfrak{B}^*$ or $\Sigma_1\,cf\,\mathfrak{B}$ is not a successor cardinal in $\mathfrak{B}$). Then there are $\mathfrak{B}$-r.e. sets A,B which are not $\mathfrak{B}$-recursive such that for every $D\subseteqq\beta$:

$D\leqq_{w\mathscr{B}}A$ and $D\leqq_{w\mathscr{B}}B\Rightarrow D$ is $\mathfrak{B}$-recursive.

Proof. Take a minimal pair of hyperregular r.e. sets A,B in $\mathfrak{A}$ and $\mathfrak{A}$-r.e. β-immune sets $\tilde{A},\tilde{B}$ of the same $\mathfrak{A}$-degree. Then $\tilde{A},\tilde{B}$ is again a hyperregular minimal pair in $\mathfrak{A}$ and it is a minimal pair in $\mathfrak{B}$ as well: Assume that for some $D\subseteqq\beta: D\leqq_{w\mathscr{B}}\tilde{A}$ and $D\leqq_{w\mathscr{B}}\tilde{B}$. Then $P[D]\leqq_{w\mathscr{B}}\tilde{A}$ and $P[D]\leqq_{w\mathscr{B}}\tilde{B}$ (P is the one-one onto $\Sigma_1\mathfrak{B}$ projection $\beta\to\kappa$). Since $\tilde{A},\tilde{B}$ are β-immune we write $\leqq_{w\mathscr{A}}$ instead of $\leqq_{w\mathscr{B}}$. Therefore $P[D]$ is $\Delta_1\mathfrak{A}$ which implies that D is $\Delta_1\mathfrak{B}$.

Theorem 8. Let $\mathfrak{B}=\langle J_\gamma,C\rangle$ be weakly inadmissible and assume that $cf2\mathfrak{B}\geqq p2\mathfrak{B}$. Then there exists a set B which is not $\mathfrak{B}$-recursive but $\mathfrak{B}$-recursive in the complete $\mathfrak{B}$-r.e. set such that for every $D\subseteqq\beta$ the following holds: $D\leqq_{w\mathscr{B}}B\Rightarrow B\leqq_{\mathscr{B}}D$ or D is $\mathfrak{B}$-recursive.

Proof. Take a complete $\mathfrak{A}$-r.e. A which is regular over $\mathfrak{A}$ ($\mathfrak{A}$ is again the admissible collaps). Construct $B\subseteqq\kappa$ in the structure $\langle\mathfrak{A},A\rangle$ as in Theorem 6 (we have $cf2\mathfrak{B}$

$= cf\,2\mathfrak{A}$ and $p2\mathfrak{B} = p2\mathfrak{A}$). Observe that B has in fact the property that for each $D \subsetneq \kappa$, $D \leqq_{w\mathscr{A}} B$ *either* $B \leqq_{\mathscr{A}} D$ or D is $\mathfrak{A}$-recursive. Then the β-immune set $\tilde{B} \subsetneq \kappa$ with $\tilde{B} =_{\mathscr{A}} B$ has the desired properties. We have $\tilde{B} \leqq_{\mathscr{A}} A$ and therefore $\tilde{B} \leqq_{\mathscr{B}} A$ because B is β-immune. Further A is $\Sigma_1 \mathfrak{A}$ therefore $\Sigma_1 \mathfrak{B}$, and so A is $\mathfrak{B}$-recursive in the universal $\mathfrak{B}$-r.e. C. This implies that $\tilde{B} \leqq_{\mathscr{B}} C$. For the completion of the proof one proceeds as before, using the fact that $\tilde{B} \leqq_{\mathscr{A}} P[D]$ implies $\tilde{B} \leqq_{\mathscr{B}} D$.

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Added in proof:

1) Sukonick constructed in his thesis (MIT, 1969) a hyperregular minimal pair for the special case $\alpha^* = \omega$.
2) Stronger versions of Theorem 7 and Theorem 8 are available by using results from [8] (see [8], Remark 7).