Annals of Mathematical Logic 16 (1979) 205–231

ON α- AND β-RECURSIVELY ENUMERABLE DEGREES

Wolfgang MAASS

Mathematisches Institut der Universität München, West Germany

Received 4 April 1978

Several problems in recursion theory on admissible ordinals (α-recursion theory) and recursion theory on inadmissible ordinals (β-recursion theory) are studied. Fruitful interactions between both theories are stressed. In the first part the admissible collapse is used in order to characterize for some inadmissible β the structure of all β-recursively enumerable degrees as an accumulation of structures of $\mathfrak{A}$-recursively enumerable degrees for many admissible structures $\mathfrak{A}$. Thus problems about the β-recursively enumerable degrees can be solved by considering "locally" the analogous problem in an admissible $\mathfrak{A}$ (where results of α-recursion theory apply). In the second part β-recursion theory is used as a tool in infinite injury priority constructions for some particularly interesting α (e.g. ω_1^{CK}). New effects can be observed since some structure of the inadmissible world above O' is projected into the α-recursively enumerable degrees by inverting the jump. The gained understanding of the jump of α-recursively enumerable degrees makes it possible to solve some open problems.

A few years ago S.D. Friedman and G.E. Sacks [1] started a new chapter in generalized recursion theory: β-recursion theory. So far recursion theory was studied only on those initial segments L_α of the constructible hierarchy where α is admissible. In β-recursion theory one considers initial segments L_β for any limit ordinal β. This is a natural step since the concept of a recursively enumerable (r.e.) set does not require any closure condition for the considered universe.

The β-r.e. sets are defined to be those subsets of L_β which are Σ_1-definable over L_β. A function is β-recursive if its graph is β-r.e. Another important concept of recursion theory is finiteness and as in α-recursion theory a subset of L_β is called β-finite if it is an element of L_β.

A striking new effect in β-recursion theory is the appearance of β-finite sets which are rather large (compared with the whole universe L_β). If β is inadmissible then there exist β-recursive functions which are cofinal in β and which have as domain a β-finite set. The minimal $\gamma<\beta$ which occurs as domain of such a function is called the recursive cofinality of β ($\sigma 1 \operatorname{cf} \beta$). This ordinal is a good measure for the remaining "admissibility" of an inadmissible β. Only β-finite sets of cardinality less than $\sigma 1 \operatorname{cf} \beta$ (in L_β) behave like α-finite sets.

The preceding example shows already that several elementary facts from ordinary recursion theory do not remain true in β-recursion theory. But usually facts of this kind are not considered to be the essential results of recursion theory. Thus the question arises what is happening e.g. with the structure of recursively enumerable degrees. Serious mathematical problems occur here since most of the constructions from α-recursion theory rely on admissibility and one has to look for new strategies.

In Section 1 of this paper we study β-recursively enumerable degrees for weakly inadmissible β and continue our earlier paper [5]. An inadmissible β is called weakly inadmissible if one can project β β-recursively into $\sigma 1\ \mathrm{cf}\ \beta$, it is called strongly inadmissible otherwise. For weakly inadmissible β one takes a suitable predicate $\tilde{T} \subseteq L_\beta$ which encodes all Δ_0-facts about L_β. One projects $\tilde{T}$ by means of the existing β-recursive projection into $\sigma 1\ \mathrm{cf}\ \beta$. We write T for the projection of $\tilde{T}$ and call the structure $\mathfrak{A} := \langle L_{\sigma 1\ \mathrm{cf}\ \beta}, T\rangle$ the admissible collapse of β. Then every set $A \subseteq L_{\sigma 1\ \mathrm{cf}\ \beta}$ is Σ_1-definable over L_β iff it is Σ_1-definable over $\mathfrak{A}$. Further $\mathfrak{A}$ is admissible and therefore we know a lot about $\mathfrak{A}$-degrees since most of the numerous results in α-recursion theory remain true for such an admissible structure with an additional regular predicate (e.g. Shore's density theorem [10] holds for $\mathfrak{A}$). It was shown in [5] that the β-recursive degrees (together with $\leq_\beta$) are isomorphic to the $\mathfrak{A}$-r.e. degrees (together with $\leq_{\mathfrak{A}}$). The occurrence of nonzero β-recursive degrees is a typical phenomenon of inadmissible recursion theory. It is easy to see that the greatest β-recursive degree lies strictly between 0 — the degree of the empty set — and the greatest β-r.e. degree $0'$. But nothing else is known about nonrecursive r.e. degrees in β-recursion theory. Theorem 1 throws some light onto this problem. A careful analysis shows that we can in fact extend the isomorphism from [5] onto all regular β-r.e. degrees (a set $A \subseteq L_\beta$ is regular if $\forall \gamma < \beta\ (A \cap L_\gamma \in L_\beta)$, a β-r.e. degree is regular if it contains a regular β-r.e. set). As isomorphic images we get in $\mathfrak{A}$ those degrees which are tame r.e. in an $\mathfrak{A}$-r.e. degree below it. This leads to the characterization of the structure of all regular β-r.e. degrees as an accumulation of structures of $\mathfrak{B}$-r.e. degrees for many different admissible $\mathfrak{B}$ (see the conclusions following Theorem 1). In particular by applying the splitting theorem to many local structures we get a global splitting theorem for all regular β-r.e. degrees.

Theorem 1 contains in addition a complete characterization of those β-r.e. degrees which are regular. The relative size of $\sigma 2\ \mathrm{cf}\ \beta$ and $\sigma 2 p\ \beta$ turns out to be the decisive criterion — a criterion which is well known from α-recursion theory (see [6] and Shore [12]). In particular for weakly inadmissible β with $\sigma 2\ \mathrm{cf}\ \beta \geq \sigma 2 p\ \beta$ the regular set theorem from α-recursion theory holds: every β-r.e. degree is regular. Therefore for these β we get an oversight over all β-r.e. degrees from the isomorphism result.

It should be mentioned that Sack's concept of tameness plays a crucial role in the formulation and the proof of Theorem 1 ($A \subseteq L_\beta$ is defined to be β-tame r.e. if $\{K \in L_\beta \mid K \subseteq A\}$ is β-r.e.).

Using the result from Theorem 1 we prove in Theorem 2 that the β-recursive degrees are not an initial segment of the β-r.e. degrees for weakly inadmissible β. We do this by performing a suitable construction in the admissible collapse of β.

In Section 2 of the paper we apply results from β-recursion theory in α-recursion theory. If α is an admissible ordinal with $\alpha > \sigma 2\ \mathrm{cf}\ \alpha \geq \sigma 2 p\ \alpha$, then the structure of the $\Sigma_2 L_\alpha$ degrees between $0'$ and $0''$ is isomorphic to the structure of the $\mathfrak{B}$-r.e. degrees for the weakly inadmissible structure $\mathfrak{B} := \langle L_\alpha, C\rangle$ (with a regular α-r.e. predicate $C \in 0'$). Thus we can apply the preceding results and we find a strange new world of α-degrees between $0'$ and $0''$ (observe that the case of metarecursion theory is included since ω_1^{CK} satisfies the condition $\alpha > \sigma 2 \operatorname{cf} \alpha \geq \sigma 2 p\, \alpha$).

The considered α are of particular interest with respect to the jump because these are the only α where the jumps of α-r.e. degrees are not yet known (for Σ_2-admissible α the situation is exactly as in ordinary recursion theory according to Theorem 5, in the case $\sigma 2 \operatorname{cf} \alpha < \sigma 2 p\, \alpha$ the jump is completely distorted and only the degrees $0'$, $0^{\frac{3}{2}}$ and $0''$ can possibly occur as jumps of α-r.e. degrees according to [6]). The distinguished degree $0^{\frac{3}{2}}$ was described in [6] for those α where incomplete non-hyperregular α-r.e. degrees exist. Observe that these degrees do always exist in the considered case $\alpha > \sigma 2 \operatorname{cf} \alpha \geq \sigma 2 p\, \alpha$ (an α-r.e. degree $\boldsymbol{a}$ is hyperregular iff $\langle L_\alpha, A \rangle$ is admissible for regular $A \in \boldsymbol{a}$). It is easy to see that every low degree is hyperregular and every high degree is non-hyperregular ($\boldsymbol{a}$ is low if $\boldsymbol{a}' = 0'$ and $\boldsymbol{a}$ is high if $\boldsymbol{a}' = 0''$). We show that there are differences among the hyperregular and among the non-hyperregular α-r.e. degrees since not every hyperregular α-r.e. degree is low and not every non-hyperregular α-r.e. degree is high.

It is easy to see that $\boldsymbol{a}' \leq 0^{\frac{3}{2}}$ for hyperregular and $\boldsymbol{a}' \geq 0^{\frac{3}{2}}$ for non-hyperregular α-r.e. degrees $\boldsymbol{a}$. Theorem 3 shows that there exist in fact non-hyperregular α-r.e. degrees $\boldsymbol{a}$ such that $\boldsymbol{a}' = 0^{\frac{3}{2}}$. Together with results from [6] we thus get

$$0^{\frac{3}{2}} = \inf \{\boldsymbol{a}' \mid \boldsymbol{a} \text{ is a non-hyperregular } \alpha\text{-r.e. degree}\}$$

for all α where incomplete non-hyperregular α-r.e. degrees exists so that we have another characterization of the degree $0^{\frac{3}{2}}$. The proof of Theorem 3 is based on a simple trick since the straightforward approach fails. We construct an α-r.e. set A as if we want to make it both non-hyperregular and low. This is of course impossible but the constructed non-hyperregular set A is then at least "as low as possible". Thus we get $A' = 0^{\frac{3}{2}}$.

Concerning the jump of hyperregular degrees we first observe that there is a rich structure of tame $\Sigma_2 L_\alpha$ (or equivalently $\Delta_2 L_\alpha$) degrees between $0'$ and $0^{\frac{3}{2}}$ for the considered α (A is tame $\Sigma_2 L_\alpha$ if $\{K \in L_\alpha \mid K \subseteq A\}$ is $\Sigma_2 L_\alpha$). These α-degrees are isomorphic to the $\mathfrak{A}$-r.e. degrees of an admissible structure $\mathfrak{A}$. We show in Theorem 4 that each of these tame $\Sigma_2 L_\alpha$ degrees is the jump of a hyperregular α-r.e. degree. This holds in particular for the greatest tame $\Sigma_2 L$ degree $0^{\frac{3}{2}}$. Thus $0^{\frac{3}{2}}$ is the only point (for any α) where the jump of a hyperregular and a non-hyperregular α-r.e. degree meet together.

We learn from the preceding results that the inadmissible world above $0'$ casts its shadow upon the central part of α-recursion theory: the structure of α-r.e. degrees and sets. In particular we stumble upon the naturally arising notion of an intermediate degree which is characterized by the property $\boldsymbol{a}' = 0^{\frac{3}{2}}$. Essential differences between the structure of r.e. degrees in ordinary recursion theory and the structure of α-r.e. degrees for some α with $\sigma 2 \operatorname{cf} \alpha < \sigma 2 p\, \alpha$ have already been discovered by R. Shore [12]. A further investigation of intermediate degrees may show several differences between ordinary recursion theory and α-recursion theory for some α with $\alpha > \sigma 2 \operatorname{cf} \alpha \geq \sigma 2 p\, \alpha$ (including metarecursion theory). By combining recent work of A. Leggett [3] with our results we can give a first example: Martin's celebrated Theorem ("a r.e. degree contains a maximal set iff it is high") is not true in metarecursion theory since for ω_1^{CK} there exist maximal sets of intermediate degree.

The combination of Leggett's results with Theorem 4 settles in addition a conjecture from Simpson's thesis [13] positively: There exist in fact hyperregular maximal sets even in nontrivial cases (it turns out that all these sets are of intermediate degree).

In the proof of Theorem 4 we use similar infinite injury strategies as in [6]. But the basic ingredient of the construction is a regular set theorem from β-recursion theory [5] (in fact this is the first application of a regular set theorem from β-recursion theory). All attempts to prove this regular set theorem by using standard methods of α-recursion theory did fail so far.

Finally in Theorem 5 we show that an admissible α is Σ_2-admissible iff it satisfies Sack's jump theorem ("every Σ_2 degree between $0'$ and $0''$ is the jump of an incomplete r.e. degree"). The crucial point of the proof is the demonstration of the failure of Sack's theorem in the case $\alpha > \sigma 2 \operatorname{cf} \alpha \geq \sigma 2p\, \alpha$. We apply here another nontrivial result from β-recursion theory (the preceding Theorem 2).

0. Preliminaries

We use the same notations and definitions as in our preceding papers [5] and [6]. All missing definitions can be found there.

It is not relevant for our arguments whether one chooses L_β or S_β (see [5]) as the universe for β-recursion theory.

For any structure $\mathfrak{B} = \langle L_\beta, B\rangle$ with $B \subseteq L_\beta$ we say that a subset of L_β is $\mathfrak{B}$-r.e. ($\mathfrak{B}$-recursive) if it is $\Sigma_1\mathfrak{B}$ ($\Delta_1\mathfrak{B}$). We write $\sigma n \operatorname{cf} \mathfrak{B}$ for the least $\gamma \leq \beta$ such that a cofinal $\Sigma_n\mathfrak{B}$ function $f:\gamma \to \beta$ exists. $\sigma np\, \mathfrak{B}$ is the least $\gamma \leq \beta$ such that an 1–1 $\Sigma_n\mathfrak{B}$ function $f:\beta\to\gamma$ exists. For the special case $\mathfrak{B} = L_\beta$ we write $\sigma n \operatorname{cf} \beta$ respectively $\sigma np\, \beta$.

Define for any structure $\mathfrak{B} = \langle L_\beta, B\rangle$ $\rho^{\mathfrak{B}}_{n,\beta} := \mu\delta \leq \beta$ (a $\Sigma_n\mathfrak{B}$ *set* $M \subseteq \delta$ exists such that $M \notin L_\beta$).

The greatest $\mathfrak{B}$-recursive degree is always denoted by $\boldsymbol{r}$.

If $\mathfrak{B}$ is a weakly inadmissible structure (i.e. $\beta > \sigma 1 \operatorname{cf} \mathfrak{B} \geq \sigma 1p\, \mathfrak{B}$) we reserve the letter κ for $\sigma 1 \operatorname{cf} \mathfrak{B}$ and we write $\mathfrak{A}$ for the admissible collapse $\langle L_\kappa, T\rangle$ of $\mathfrak{B}$. A set $A \subseteq L_\kappa$ is called β-immune if it is immune with respect to neighborhood conditions out of $L_\beta - L_\kappa$, i.e. for any $K \in L_\beta$ we have $(K \subseteq A \vee K \subseteq L_\kappa - A) \to K \in L_\kappa$. According to [5] one can define for every set $A \subseteq L_\kappa$ a β-immune version $\tilde{A} \subseteq L_\kappa$ of the same $\mathfrak{A}$-degree. The operation $\tilde{}$ preserves regularity and $\Sigma_n\mathfrak{A}$-definability.

For sets $A, B \subseteq \beta$ we set $A \vee B := 2A \cup 2B+1$ where $2A := \{2x \mid x \in A\}$ and $2B+1 := \{2x+1 \mid x \in B\}$.

Finally we remind of two conventions. If we write $L_\gamma \models [x \in W]$ for some β-r.e. set W, then this means that we have fixed a $\Sigma_1 L_\beta$ definition ϕ of W and $L_\gamma \models \phi(x)$.

Further if we write $L_\gamma \models \psi$ for a formula ψ, then this implies that every parameter in ψ is an element of L_γ.

1. β-recursively enumerable degrees

Theorem 1. *Assume β is weakly inadmissible and $\boldsymbol{b}$ is a β-degree. Then* (1), (2), (3) *are equivalent:*

(1) *$\boldsymbol{b}$ contains a regular β-r.e. set,*

(2) *$\boldsymbol{b}$ contains a β-r.e. set and $\neg(\boldsymbol{r}<\boldsymbol{b} \wedge \sigma 2\,\mathrm{cf}\,\beta < \sigma 2p\,\beta)$ ($\boldsymbol{r}$ is the greatest β-recursive degree),*

(3) *$\boldsymbol{b}$ contains a β-immune set $A \subseteq L_\kappa$ such that A is regular over L_κ and $\langle\mathfrak{A}, B\rangle$-tame r.e. for some $\mathfrak{A}$-r.e. set $B \subseteq L_\kappa$ with $B \leqslant_{\mathfrak{A}} A$.*

Proof. (2)→(3). Assume (2) and take a β-r.e. set $H \in \boldsymbol{b}$. We will construct β-immune sets $B, \hat{A} \subseteq \kappa$ such that $H =_\beta \hat{A}$, $\langle\mathfrak{A}, B\rangle$ is not strongly inadmissible, B is regular over L_κ, $B \leqslant_{\mathfrak{A}} \hat{A}$ and $\hat{A}$ is $\langle\mathfrak{A}, B\rangle$-tame r.e. This is enough in order to show (3) because we can apply then the regular set theorem for tame r.e. sets in weakly inadmissible structures ((1)↔(3) of Theorem 4 in [5]) respectively the usual regular set theorem for r.e. sets in admissible structures to the $\langle\mathfrak{A}, B\rangle$-tame r.e. set $\hat{A}$ in the weakly inadmissible or admissible structure $\langle\mathfrak{A}, B\rangle$. This gives a $\langle\mathfrak{A}, B\rangle$-tame r.e. set A^* with A^* regular over L_κ and $\hat{A} =_{\langle\mathfrak{A},B\rangle} A^*$. Then we take the β-immune set $\tilde{A}^*$ and define $A := B \vee \tilde{A}^*$. It is easy to verify that B, A have all the properties which are demanded in (3).

We construct now the sets $B, \hat{A}$ with the properties above (i.e. all properties from (3) except $\hat{A}$ regular).

Fix a β-recursive function P which maps L_β 1–1 onto κ. Define $B_1 := P[H]$. B_1 is $\mathfrak{A}$-r.e. and there is a regular $\mathfrak{A}$-r.e. set B_2 such that $B_1 =_{\mathfrak{A}} B_2$. Then we define $B := \tilde{B}_2$. It is obvious that $B =_{\mathfrak{A}} B_2 =_{\mathfrak{A}} B_1$ and B is $\mathfrak{A}$-r.e., regular and β-immune. We further have $B \leqslant_\beta H$ by the construction of B_1 since B is β-immune and $B =_{\mathfrak{A}} B_1$.

Define $\hat{\mathfrak{A}} := \langle\mathfrak{A}, B\rangle$. We want to show that $\hat{\mathfrak{A}}$ is not strongly inadmissible and at this point we use the assumption $\neg(\boldsymbol{r}<\boldsymbol{b} \wedge \sigma 2\,\mathrm{cf}\,\beta < \sigma 2p\,\beta)$ in (2).

(a) *B is incomplete in $\mathfrak{A}$.* According to Lemma 3.3 in Shore [10] we have $\sigma 1\,\mathrm{cf}^{\hat{\mathfrak{A}}}\,\kappa \geqslant \rho^{\hat{\mathfrak{A}}}_{1,\kappa}$. Further by using the fact that $\sigma 2p^{\mathfrak{A}}\,\kappa = \rho^{\mathfrak{A}}_{2,\kappa}$ we get $\rho^{\hat{\mathfrak{A}}}_{1,\kappa} \geqslant \sigma 1p^{\hat{\mathfrak{A}}}\,\kappa$ according to Shore [10]. The used equality holds because we have $\sigma 2p^{\mathfrak{A}}\,\kappa = \sigma 2p\,\beta = \rho_{2,\beta} = \rho^{\mathfrak{A}}_{2,\kappa}$. Thus it is proved that $\sigma 1\,\mathrm{cf}^{\hat{\mathfrak{A}}}\,\kappa \geqslant \sigma 1p^{\hat{\mathfrak{A}}}\,\kappa$.

(b) *B is complete in $\mathfrak{A}$.* In this case B is an element of the β-degree $\boldsymbol{r}$ according to Theorem 8 in [5]. We know already that $B \leqslant_\beta H$ and therefore we have $\boldsymbol{r} \leqslant \boldsymbol{b}$. Since (3) follows from Theorem 4 in [5] if $\boldsymbol{r} = \boldsymbol{b}$ we assume $\boldsymbol{r} < \boldsymbol{b}$ for the following. This implies $\sigma 2\,\mathrm{cf}\,\beta \geqslant \sigma 2p\,\beta$ according to the assumption in (2). Therefore we have $\sigma 1\,\mathrm{cf}^{\hat{\mathfrak{A}}}\,\kappa = \sigma 2\,\mathrm{cf}\,\beta \geqslant \sigma 2p\,\beta = \sigma 1p^{\hat{\mathfrak{A}}}\,\kappa$.

Since $\hat{\mathfrak{A}}$ is not strongly inadmissible there is a $\hat{\mathfrak{A}}$-recursive function $\hat{P}$ which maps κ 1–1 onto $\hat{\kappa} := \sigma 1\,\mathrm{cf}^{\hat{\mathfrak{A}}}\,\kappa$. Further we fix an $\hat{\mathfrak{A}}$-recursive strictly increasing cofinal function $\hat{q}:\hat{\kappa}\to\kappa$.

We define

$$A_1:=\{\langle 0,x\rangle\in\kappa \mid P^{-1}(x)\cap H\neq\emptyset\}\cup\{\langle 1,x\rangle\in\kappa \mid P^{-1}(x)\cap L_\beta - H\neq\emptyset\},$$

$$A_2:=\hat{q}[\hat{P}[A_1]],\qquad A_3:=\tilde{A}_2\quad\text{and}\quad \hat{A}:=B\vee A_3.$$

Then $\hat{A}$ is β-immune by construction and it is obvious that $B\leqslant_{\mathfrak{A}}\hat{A}$. In order to verify the other properties of $\hat{A}$ we prove:

Claim. Assume that $K\in L_\kappa$, κ-card$(K)<\hat{\kappa}$ and $K\subseteq A_1\cap\{1\}\times\kappa$. Then there exists a set $K^*\in L_\kappa$ such that

$$\kappa\text{-card}(K^*)<\hat{\kappa},\qquad P^{-1}[K^*]\subseteq L_\beta - H$$

and

$$\forall\langle 1,x\rangle\in K\quad \exists z\in P^{-1}[K^*](z\in P^{-1}(x))$$

(thus K^* is a "small" κ-finite set of witnesses for "$K\subseteq A_1\cap\{1\}\times\kappa$").

Proof of the claim. We have $P[H]=B_1=_{\mathfrak{A}}B$. Therefore we can write

$$\text{“}y\in\kappa - P[H]\wedge P^{-1}(y)\in P^{-1}(x)\text{”}$$

as a $\Sigma_1\mathfrak{A}$ formula $\psi(x,y)$. Since $K\subseteq A_1\cap\{1\}\times\kappa$ we have that

$$\forall x\in K\quad \exists y\in\kappa(\mathfrak{A}\models\psi(x,y)).$$

Since κ-card$(K)<\hat{\kappa}$ there exists in fact a κ-finite function $h:K\to\kappa$ such that $\forall x\in K(\mathfrak{A}\models\psi(x,h(x)))$ (show first that the function $g:K\to\kappa$ such that

$$\forall x\in K(g(x)=\mu\delta(\langle L_\delta, T\cap L_\delta, B\cap L_\delta\rangle\models[\exists y\psi(x,y)])$$

is κ-finite because κ-card$(K)<\hat{\kappa}=\sigma 1\ \mathrm{cf}^{\mathfrak{A}}\,\kappa$).

Then the set $K^*:=\mathrm{Rg}\,h$ has all the properties which are demanded in the claim. □

We can show now that $\hat{A}$ is $\mathfrak{A}$-tame r.e.: We have

$$K\in L_\kappa\wedge K\subseteq\hat{A}\leftrightarrow\exists\hat{K}\in L_\kappa(\langle K,\hat{K}\rangle\in W_e\wedge\kappa\text{-card}(\hat{K})<\hat{\kappa}\wedge\hat{K}\subseteq A_1\cap\{1\}\times\kappa)$$

for some suitable $\mathfrak{A}$-r.e. set W_e. By the claim we further have

$$\begin{aligned}&\hat{K}\in L_\kappa\wedge\kappa\text{-card}(\hat{K})<\hat{\kappa}\wedge\hat{K}\subseteq A_1\cap\{1\}\times\kappa\\&\leftrightarrow\exists K^*\in L_\kappa(\kappa\text{-card}(K^*)<\hat{\kappa}\wedge P^{-1}[K^*]\subseteq L_\beta - H\\&\qquad\qquad\wedge\forall\langle 1,x\rangle\in\hat{K}\quad\exists z\in P^{-1}[K^*](z\in P^{-1}(x))).\end{aligned}$$

The latter can be written as a $\Sigma_1\mathfrak{A}$ formula since

$$P^{-1}[K^*]\subseteq L_\beta - H\leftrightarrow K^*\subseteq L_\kappa - B_1$$

and $B_1\leqslant_{\mathfrak{A}}B$. Thus we have shown that $\hat{A}$ is $\mathfrak{A}$-tame r.e.

$H \leq_\beta \hat{A}$ follows from

$$K \in L_\beta \wedge K \subseteq H \leftrightarrow \exists x \in \kappa(P^{-1}(x) = K \wedge \langle 1, x\rangle \notin A_1),$$

$$K \in L_\beta \wedge K \subseteq L_\beta - H \leftrightarrow \exists x \in \kappa(P^{-1}(x) = K \wedge \langle 0, x\rangle \notin A_1)$$

and $A_1 \leq_{w\mathfrak{A}} \hat{A}$ (it is obvious that $A_1 \leq_{w\hat{\mathfrak{A}}} \hat{A}$; this implies $A_1 \leq_{w\mathfrak{A}} \hat{A}$ because $B \leq_{\mathfrak{A}} \hat{A}$).

It only remains to show that $\hat{A} \leq_\beta H$. We have

$$K \in L_\kappa \wedge K \subseteq A_2 \leftrightarrow K \in L_\kappa \wedge \exists \hat{K} \in L_\kappa(K = \hat{q}[\hat{P}[\hat{K}]] \wedge \kappa\text{-card}\,(\hat{K}) < \hat{\kappa})$$
$$\wedge \exists K_0 K_1(K = K_0 \cup K_1 \wedge K_0 \subseteq \{0\} \times \kappa \cap A_1 \wedge K_1 \subseteq \{1\} \times \kappa \cap A_1).$$

The part $K_0 \subseteq \{0\} \times \kappa \cap A_1$ is obviously $\Sigma_1 L_\beta$ since β-card$\,(K_0) < \kappa$. Further we have shown before that for κ-finite sets K_1 of κ-cardinality less than $\hat{\kappa}$ we can write $K_1 \subseteq \{1\} \times \kappa \cap A_1$ as a $\Sigma_1 \hat{\mathfrak{A}}$ formula. Since $B \leq_\beta H$ we have altogether expressed $K \in L_\kappa \wedge K \subseteq A_2$ β-recursively in H. Since $\hat{A}$ is β-immune we can therefore express $K \in L_\beta \wedge K \subseteq \hat{A}$ β-recursively in H.

In order to show the other part of $\hat{A} \leq_\beta H$ we observe that

$$K \in L_\kappa \wedge K \subseteq \kappa - A_1 \leftrightarrow \exists K_1 K_2 \in L_\beta(K_1 = \bigcup\{\check{K} \mid \langle 0, P(\check{K})\rangle \in K\}$$
$$\wedge K_1 \subseteq L_\beta - H \wedge K_2 = \bigcup\{\check{K} \mid \langle 1, P(\check{K})\rangle \in K\} \wedge K_2 \subseteq H).$$

We further have

$$K \in L_\kappa \wedge K \subseteq L_\kappa - A_2 \leftrightarrow \exists \hat{K} \in L_\kappa(\hat{q}[\hat{P}[\hat{K}]] = \mathrm{Rg}\, \hat{q} \cap K \wedge \hat{K} \subseteq \kappa - A_1).$$

The right side of this equivalence can be written as a $\Sigma_1 \hat{\mathfrak{A}}$ formula since $\hat{q}$ is strictly increasing and continuous. If we combine these facts it is easy to see that $K \in L_\beta \wedge K \subseteq L_\beta - \hat{A}$ can be expressed β-recursively in H.

(3)$\to$(1). Take sets A, B according to (3). One can assume that B is in addition β-immune and regular over L_κ. We are going to define a $\Pi_1 L_\beta$ set H which is regular over L_β such that $A =_\beta H$. Then $L_\beta - H$ will be a regular β-r.e. set of the same β-degree as A.

One might try to understand the definition of H as follows: There exists some $\Pi_1 \mathfrak{A}$ set D such that $w \in A \leftrightarrow \exists z \in L_\kappa(\langle w, z\rangle \in D)$. H is some sort of $\Pi_1 \mathfrak{A}$-uniformization of this relation D.

First we define a set $\hat{H}$. Fix a $\Sigma_1 \langle \mathfrak{A}, B\rangle$ formula ϕ such that $R \subseteq A \leftrightarrow \langle \mathfrak{A}, B\rangle \models \phi(R)$. For $w \in A$ we put a 4-tupel $\langle w, \gamma, \delta, K\rangle$ into $\hat{H}$ such that

(a) γ is minimal such that

$$\langle L_\gamma, T \cap L_\gamma, B \cap L_\gamma\rangle \models \exists R(w \in R \wedge \phi(R)),$$

(b) $K = B \cap L_\gamma$,

(c) $\delta \geq \gamma$ is minimal such that

$$\forall x \in K(\langle L_\delta, T \cap L_\delta\rangle \models \psi(x))$$

where ψ is some fixed $\Sigma_1 \mathfrak{A}$ definition of B.

For every $w \in A$ there exists exactly one tripel $\langle \gamma, \delta, K\rangle$ such that $\langle w, \gamma, \delta, K\rangle \in \hat{H}$ and $\hat{H}$ is $\Pi_1 \mathfrak{A}$ definable.

Fix a β-finite function U which maps L_κ 1–1 onto κ and a β-recursive strictly increasing and continuous cofinal function q from κ into β. We write $\tilde{q}$ for $q \circ U$.

Define the set H as follows (we use set theoretic pairing):

$$\langle x, \gamma, \delta, K\rangle \in H :\leftrightarrow x \in \tilde{q}[L_\kappa] \wedge \langle \tilde{q}^{-1}(x), \gamma, \delta, K\rangle \in \hat{H}.$$

H is $\Pi_1 L_\beta$ definable since $\tilde{q}[L_\kappa]$ is $\Delta_1 L_\beta$ and $\hat{H}$ is $\Pi_1 L_\beta$.

Take some set $R_0 \in L_\kappa$ such that $R_0 \subseteq A$. Then there exists some $\gamma_0 < \kappa$ such that for all $w \in R_0$

$$\langle L_{\gamma_0}, T \cap L_{\gamma_0}, B \cap L_{\gamma_0}\rangle \models \exists R(w \in R \wedge \phi(R))$$

(simply choose γ_0 such that $R_0 \in L_{\gamma_0}$ and $\langle L_{\gamma_0}, T \cap L_{\gamma_0}, B \cap L_{\gamma_0}\rangle \models \phi(R_0)$). It follows from this observation that $\hat{H} \restriction R_0$ is β-finite. We further have that $\tilde{q} \restriction R_0$ is β-finite and therefore $H \restriction \tilde{q}[R_0]$ is β-finite as well. Observe that the given argument relies mainly on the fact that A is $\langle \mathfrak{A}, B\rangle$-*tame* r.e.

We can now prove that H is regular over L_β. Assume that some γ such that $\kappa \leq \gamma < \beta$ is given. In order to show that $L_\gamma \cap H \in L_\beta$ we define

$$K := \{w \in L_\kappa \mid \exists y_1 y_2 y_3 \in L_\kappa(\langle \tilde{q}(w), y_1, y_2, y_3\rangle \in L_\gamma)\}.$$

By using the properties of $\tilde{q}$ we see that $K \in L_\kappa$. Since A is regular over L_κ we have $R := K \cap A \in L_\kappa$. By the preceding we have then $H \restriction \tilde{q}[R] \in L_\beta$. Since $H \cap L_\gamma = H \restriction \tilde{q}[R] \cap L_\gamma$ this implies that $H \cap L_\gamma \in L_\beta$.

$A \leq_\beta H$. We have

$$R \in L_\beta \wedge R \subseteq A \leftrightarrow R \in L_\kappa \wedge \exists F \in L_\beta((F \text{ is a function}) \wedge \operatorname{dom} F = \tilde{q}[R] \wedge F \subseteq H)$$

and

$$R \in L_\beta \wedge R \subseteq L_\kappa - A \leftrightarrow R \in L_\kappa \wedge R \subseteq L_\kappa - A \leftrightarrow \tilde{q}[R] \times L_\kappa^3 \subseteq L_\beta - H$$

where $\tilde{q}[R] \times L_\kappa^3$ is a β-finite set due to the properties of set theoretic pairing.

$H \leq_\beta A$. We write $\pi_0, \pi_1, \pi_2, \pi_3$ for the projection functions which are associated with the set theoretic 4-tuples. We have:

$$\begin{aligned}
F \in L_\beta \wedge F \subseteq H \leftrightarrow{} & F \in L_\beta \wedge (F \text{ is a function}) \wedge \exists R \in L_\kappa(\operatorname{dom} F \\
&= \tilde{q}[R] \wedge R \subseteq A \wedge \exists \gamma\delta K \in L_\kappa(\gamma \leq \delta < \kappa \wedge \pi_1[F] \subseteq \gamma \wedge K \subseteq L_\gamma \\
&\wedge L_\gamma - K \subseteq L_\gamma - B \wedge \exists K_1(K_1 = L_\delta \cap T \wedge \forall x \in K(\langle L_\delta, K_1\rangle \models \psi(x)) \\
&\wedge \pi_2[F] \subseteq \delta \wedge \forall \hat{K} \in \pi_3[F] \exists \hat{\gamma} < \gamma(\hat{K} = K \cap L_{\hat{\gamma}}) \wedge \exists \rho < \beta(\kappa < \rho \\
&\wedge F \in L_\rho \wedge \tilde{q} \restriction R \in L_\rho \wedge L_\rho \models [F \text{ is correctly defined with respect to} \\
&\tilde{q} \restriction R, \gamma, \delta, K, K_1 \text{ according to the definition of } H])))).
\end{aligned}$$

Since T is $\Delta_1 L_\beta$ and $B \leq_\beta A$ we can express in this way $F \in L_\beta \wedge F \subseteq H$ β-recursively in A.

Concerning the other part of $H \leq_\beta A$ we have

$$\begin{aligned}
K \in L_\beta \wedge K \subseteq L_\beta - H \leftrightarrow{} & \exists K_1 K_2 K_3 F \in L_\beta(K_1 = \pi_0[K] \cap \tilde{q}[L_\kappa] \\
&\wedge K_2 \cup K_3 = \tilde{q}^{-1}[K_1] \wedge K_2 \subseteq A \wedge K_3 \subseteq L_\kappa - A \\
&\wedge \operatorname{dom} F = \tilde{q}[K_2] \wedge F \subseteq H \wedge F \cap K = \emptyset).
\end{aligned}$$

The expression "$F \subseteq H$" which occurs on the right side can be expressed β-recursively in A as we have shown before. Observe that the equivalence relies on the fact that H is a function.

(1)→(3). Assume $H \subseteq L_\beta$ is regular and β-r.e. We construct sets A, B with the properties in (3).

Fix a β-recursive function P which maps L_β 1–1 onto κ. Define $B := P[H]$. Then B is $\mathfrak{A}$-r.e. and regular over L_κ.

Define

$$A_1 := \{\langle 0, x\rangle \in \kappa \mid P^{-1}(x) \cap H \neq \emptyset\} \cup \{\langle 1, x\rangle \in \kappa \mid P^{-1}(x) \cap L_\beta - H \neq \emptyset\}$$

and $A := \tilde{A}_1$. It is obvious that $H \leqslant_\beta A$ and A is β-immune. Further the regularity of H over L_β implies the regularity of A_1 and A over L_κ.

A is $\langle \mathfrak{A}, B\rangle$-tame r.e. We have for any $K \in L_\kappa$:

$$\begin{aligned} K \subseteq A_1 \leftrightarrow \exists K_1 K_2 \in L_\kappa (K = K_1 \cup K_2 \wedge K_1 \subseteq \{0\} \times \kappa \wedge K_2 \subseteq \{1\} \times \kappa \\ \wedge \exists \gamma < \beta (L_\gamma \models [\forall \langle 0, x\rangle \in K_1\, \exists y \in P^{-1}(x)(y \in H)]) \\ \wedge \exists \hat{K} \in L_\kappa (\hat{K} \subseteq \kappa - B \wedge \forall \langle 1, x\rangle \in K_2\, \exists y \in P^{-1}[\hat{K}](y \in P^{-1}(x)))). \end{aligned}$$

Whereas "←" is obvious we have to show for "→" that the set $\hat{K} \in L_\kappa$ exists. The set $\bigcup\{P^{-1}(x) \mid \langle 1, x\rangle \in K_2\}$ is β-finite and therefore subset of some L_γ with $\gamma < \beta$. Since $L_\gamma \cap H \in L_\beta$ there exists a β-finite function h such that

$$\forall \langle 1, x\rangle \in K_1 (h(\langle 1, x\rangle) \in P^{-1}(x) \cap L_\gamma - H).$$

Then the κ-finite set $\hat{K} := P[\mathrm{Rg}\, h]$ has all the desired properties. We have thus shown that A_1 is $\langle \mathfrak{A}, B\rangle$-tame r.e. which implies that A is $\langle \mathfrak{A}, B\rangle$-tame r.e. as well.

The equivalence above proves simultaneously one part of $A \leqslant_\beta H$ if we write $P^{-1}[\hat{K}] \subseteq L_\beta - H$ instead of $\hat{K} \subseteq \kappa - B$ on the right side. The other part is immediate since A is β-immune (see the analogous reduction in the proof of (2)→(3)).

Since $H \leqslant_\beta A$ and $B = P[H]$ it is obvious that $B \leqslant_{\mathfrak{A}} A$.

(3)→(2). Assume for a contradiction that $\boldsymbol{r} < \boldsymbol{b} \wedge \sigma 2\, \mathrm{cf}\, \beta < \sigma 2p\, \beta$. Then we can choose the sets A, B according to (3) such that in addition $B <_{\mathfrak{A}} A$ and $B \in 0'$ in $\mathfrak{A}$ is regular. In this case the structure $\hat{\mathfrak{A}} := \langle \mathfrak{A}, B\rangle$ is strongly inadmissible because

$$\sigma 1\, \mathrm{cf}^{\hat{\mathfrak{A}}}\, \kappa = \sigma 2\, \mathrm{cf}^{\mathfrak{A}}\, \kappa = \sigma 2\, \mathrm{cf}\, \beta < \sigma 2p\, \beta = \sigma 2p^{\mathfrak{A}}\, \kappa = \sigma 1 p^{\hat{\mathfrak{A}}}\, \kappa.$$

We have shown in Lemma 24 in [5] that $A \leqslant_\beta \emptyset$ for every regular tame r.e. set $A \subseteq L_\beta$ if β is strongly inadmissible. The argument works as well for our strongly inadmissible *structure* $\hat{\mathfrak{A}}$ since by Jensen's Uniformization Theorem [2] we have $\sigma 2p\, \beta = \rho_{2,\beta}$ so that $\sigma 1\, \mathrm{cf}^{\hat{\mathfrak{A}}}\, \kappa < \rho_{2,\beta} = \rho^{\hat{\mathfrak{A}}}_{1,\kappa}$. This inequality $\sigma 1\, \mathrm{cf}^{\hat{\mathfrak{A}}}\, \kappa < \rho^{\hat{\mathfrak{A}}}_{1,\kappa}$ is needed for the argument. Therefore we have $A \leqslant_{\hat{\mathfrak{A}}} \emptyset$ which implies that $A \leqslant_{\mathfrak{A}} B$. This is a contradiction to the assumption $B <_{\mathfrak{A}} A$ so that we have proved that $\neg(\boldsymbol{r} < \boldsymbol{b} \wedge \sigma 2\, \mathrm{cf}\, \beta < \sigma 2p\, \beta)$ if $\boldsymbol{b}$ contains a set according to (3). It remains to show that $\boldsymbol{b}$ contains a β-r.e. set but this follows from (3)→(1).

Thus the proof of Theorem 1 is complete.

Conclusions

(1) Consider a weakly inadmissible β with admissible collapse $\mathfrak{A}$. According to Theorem 8 in [5] there exists an isomorphism I from $\langle S, \leqslant_{\mathfrak{A}}\rangle$ onto $\langle R, \leqslant_\beta\rangle$ where S is the set of $\mathfrak{A}$-r.e. $\mathfrak{A}$-degrees and R is the set of β-recursive (or equivalently β-tame r.e.) β-degrees.

Define $\hat{S}$ as the set of all $\mathfrak{A}$-degrees which contain a regular set A such that A is $\langle\mathfrak{A}, B\rangle$-tame r.e. for some $\mathfrak{A}$-r.e. set B with $B \leqslant_{\mathfrak{A}} A$. Define $\hat{R}$ as the set of all regular β-r.e. β-degrees. Then we have: There exists an isomorphism $\hat{I}$ from $\langle\hat{S}, \leqslant_{\mathfrak{A}}\rangle$ onto $\langle\hat{R}, \leqslant_\beta\rangle$ such that $\hat{I} \restriction S = I$.

The definition of $\hat{I}$ is simple: If $a \in \hat{S}$ is an $\mathfrak{A}$-degree then we take a set $A \in a$ with the properties as in the definition of $\hat{S}$ such that A is in addition β-immune (a contains such an A by the properties of the $\tilde{}$-operation in [5]). Define then $\hat{I}(a)$ as the β-degree of A. Theorem 1 implies that the so defined function $\hat{I}$ is an isomorphism.

(2) For the first time we now have an overview over the structure of all β-r.e. β-degrees for an inadmissible β. Assume β *is weakly inadmissible and* $\sigma 2 \operatorname{cf} \beta \geqslant \sigma 2p\, \beta$.

In this case the set $\hat{R}$ in the preceding conclusion is the set of all β-r.e. β-degrees. Further all structures $\langle\mathfrak{A}, B\rangle$ which occur in the definition of $\hat{S}$ for this case are not strongly inadmissible. Therefore we can drop the requirement "A regular" in the definition of $\hat{S}$ (apply the usual regular set theorem for admissible structures respectively the regular set theorem for tame r.e. sets in weakly inadmissible structures which is contained in Theorem 4 in [5]).

Thus we learn that there are many β-r.e. degrees between $\boldsymbol{r}$ and $0'$ in β: The structure of all β-r.e. degrees $\boldsymbol{b}$ such that $\boldsymbol{r} \leqslant \boldsymbol{b} \leqslant 0'$ (together with $\leqslant_\beta$) is isomorphic to the structure of all $\hat{\mathfrak{A}}$-r.e. $\hat{\mathfrak{A}}$-degrees (together with $\leqslant_{\hat{\mathfrak{A}}}$) for an admissible structure $\hat{\mathfrak{A}}$.

We get $\hat{\mathfrak{A}}$ by applying the admissible collapse two times if necessary: It is easy to see that $\hat{I}$ maps exactly the tame $\Sigma_2\mathfrak{A}$ $\mathfrak{A}$-degrees $\boldsymbol{a} \geqslant 0'_{\mathfrak{A}}$ onto the considered β-degrees $\boldsymbol{b}$, because in the definition of an $\mathfrak{A}$-degree $\boldsymbol{a} \in \hat{S}$ we can always assume that $B \in 0'$ if $\boldsymbol{a}$ satisfies $0' \leqslant \boldsymbol{a}$ ($\mathfrak{A}$ is as always the admissible collapse of β). Fix then an $\mathfrak{A}$-r.e. regular set $C \in 0'_{\mathfrak{A}}$. The structure $\langle\mathfrak{A}, C\rangle$ is either admissible, in which case we define $\hat{\mathfrak{A}} := \langle\mathfrak{A}, C\rangle$, or it is weakly inadmissible, in which case we define $\hat{\mathfrak{A}}$ to be the admissible collapse of $\langle\mathfrak{A}, C\rangle$.

In order to describe all β-r.e. degrees we write RE $(\hat{\mathfrak{A}})$ for the structure of all $\hat{\mathfrak{A}}$-r.e. degrees together with $\leqslant_{\hat{\mathfrak{A}}}$ for any admissible structure $\hat{\mathfrak{A}}$. For the considered β we can describe then the structure $\langle\hat{R}, \leqslant_\beta\rangle$ of all β-r.e. degrees as an accumulation of many structures RE $(\mathfrak{A}_{\boldsymbol{b}})$ where $\boldsymbol{b}$ ranges over the β-tame r.e. degrees. The picture we get is familiar from fireworks: Every β-tame r.e. degree $\boldsymbol{b}$ is the starting point of some structure RE $(\mathfrak{A}_{\boldsymbol{b}})$ above $\boldsymbol{b}$, where $\boldsymbol{b}$ itself corresponds to $0_{\mathfrak{A}_{\boldsymbol{b}}}$.

To be a little more exact, we start with the structure $\langle\hat{R}, \leqslant_\beta\rangle$ and go then to the isomorphic structure $\langle\hat{S}, \leqslant_{\mathfrak{A}}\rangle$. Consider a degree $\boldsymbol{a} \in \hat{S}$. Then there is a set $A \in \boldsymbol{a}$

which is $\langle\mathfrak{A}, B\rangle$-tame r.e. for some $\mathfrak{A}$-r.e. B with $B \leqslant_{\mathfrak{A}} A$. It is obvious that we can choose B in addition β-immune so that the β-degree of B is some β-tame r.e. degree $\boldsymbol{b}$. In order to define $\mathfrak{A}_{\boldsymbol{b}}$ we first observe that the structure of the $\langle\mathfrak{A}, B\rangle$-tame r.e. $\mathfrak{A}$-degrees $\boldsymbol{a} \leqslant_{\mathfrak{A}} B$ (together with $\leqslant_{\mathfrak{A}}$) is isomorphic to the structure of the $\langle\mathfrak{A}, B\rangle$-tame r.e. $\langle\mathfrak{A}, B\rangle$-degrees (together with $\leqslant_{\langle\mathfrak{A}, B\rangle}$). We define $\mathfrak{A}_{\boldsymbol{b}}$ as the structure $\langle\mathfrak{A}, B\rangle$, if $\langle\mathfrak{A}, B\rangle$ is admissible. Otherwise $\langle\mathfrak{A}, B\rangle$ is weakly inadmissible and we define $\mathfrak{A}_{\boldsymbol{b}}$ to be the admissible collapse of $\langle\mathfrak{A}, B\rangle$.

For those weakly inadmissible β which satisfy $\sigma 2 \operatorname{cf} \beta < \sigma 2p\, \beta$ one can give a similar description except that we don't know which structure has to be attached to the greatest β-tame r.e. degree $\boldsymbol{r}$ as an characterization of the β-r.e. β-degrees between $\boldsymbol{r}$ and $0'$. We expect that the degree structure of a strongly inadmissible structure occurs at this point.

Thus we see that for all β-r.e. β-degrees $\boldsymbol{d} > 0$ such that

$$\neg(\boldsymbol{r} < \boldsymbol{d} \wedge \sigma 2 \operatorname{cf} \beta < \sigma 2p\, \beta)$$

the *splitting theorem* holds: There exist β-r.e. β-degrees $\boldsymbol{d}_1, \boldsymbol{d}_2$ such that $0 < \boldsymbol{d}_1 < \boldsymbol{d}$, $0 < \boldsymbol{d}_2 < \boldsymbol{d}$ and $\boldsymbol{d}$ is the least upper bound of $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$.

The claim is immediate from [5] if $\boldsymbol{d}$ is β-tame r.e. Otherwise we apply the splitting theorem for r.e. degrees in admissible structures to the admissible structure $\mathfrak{A}_{\boldsymbol{b}}$ in which $\boldsymbol{d}$ is represented by a r.e. degree.

We do not yet know much about the "overlapping" of the structures RE $(\mathfrak{A}_{\boldsymbol{b}})$. Concerning a proof of the density theorem for regular β-r.e. degrees one can eliminate this problem[1]. The following trick in ORT is due to David Posner. Consider sets $A_1, A_2 \subseteq \omega$ such that $A_2 <_\omega A_1$ and there exist r.e. sets B_1, B_2 such that A_i is r.e. in B_i and $B_i \leqslant_\omega A_i$, $i = 1, 2$. In order to find sets A_3, B_3 such that $A_2 <_\omega A_3 <_\omega A_1$, $B_3 \leqslant_\omega A_3$ and A_3 is r.e. in B_3 one considers the following cases:

(a) $B_1 \vee A_2 =_\omega A_2$. Apply the density theorem for r.e. degrees in $\langle L_\omega, B_1 \vee B_2\rangle$.

(b) $B_1 \vee A_2 =_\omega A_1$. Then $A_2 <_\omega B_1 \vee A_2$ and $B_1 \vee A_2$ is r.e. in B_2. Apply the density theorem for r.e. degrees in $\langle L_\omega, B_2\rangle$.

(c) $A_2 <_\omega B_1 \vee A_2 <_\omega A_1$. Define $A_3 := B_1 \vee A_2$.

By combining this argument with Theorem 1 and the density theorem in α-RT (Shore [10]) one sees that the regular β-r.e. degrees are dense if β is weakly inadmissible and $\sigma 1 \operatorname{cf} \beta$ is a cardinal in L.

For a general weakly inadmissible β the problem is reduced to the density theorem for r.e. degrees in admissible structures $\langle L_\alpha, B\rangle$ with B regular. The latter problem is open, even if B is in addition α-r.e. (the proof of this case might suffice for our application). Shore's proof of the density theorem for α-r.e. degrees [10] uses properties of projecta which are dubious in presence of a predicate B.

(3) One has to be careful in generalizing the preceding results to weakly inadmissible *structures* $\mathfrak{B} = \langle L_\beta, B\rangle$. We have used that $\sigma 2p\, \beta = \rho_{2,\beta}$ (uniformization theorem [2]) at two points in the proof of Theorem 1 (in (2)→(3) and in

[1] We are grateful to Sy D. Friedman and Gerald E. Sacks who informed us about this. Further we would like to thank Richard A. Shore and the referee for pointing out the situation concerning relativized projecta.

(3)→(2)). Even for some admissible $\mathfrak{B}$ this equality is false. Fortunately the equality holds for the most interesting applications in *α-recursion theory:*

If α is admissible, $\alpha > \sigma 2 \text{ cf } \alpha \geq \sigma 2p\, \alpha$ and $B \in 0'$ is α-r.e. and regular over L_α then we have for $\mathfrak{B} := \langle L_\beta, B\rangle$ that

$$\sigma 2p^{\mathfrak{B}}\, \alpha = \sigma 3p\, \alpha = \rho_{3,\alpha} = \rho^{\mathfrak{B}}_{2,\alpha}$$

by the uniformization theorem for α. (Observe that the level 3 is the first one where we really need the full power of the uniformization theorem in α-recursion theory since Σ_2-uniformization is trivial for admissible α.) Thus we get a lot of information about $\Sigma_2 L_\alpha$ degrees above $0'$ for these α.

Theorem 2. *Assume that β is weakly inadmissible. Then the β-tame r.e. degrees are not an initial segment of the β-r.e. degrees.*

Proof. According to Theorem 1 (and Theorem 4 in [5]) it is enough to solve the following problem for the admissible collapse $\mathfrak{A} = \langle L_\kappa, T\rangle$ of L_β: Construct a set A such that $A <_{\mathfrak{A}} 0'_{\mathfrak{A}}$, A is $\langle\mathfrak{A}, B\rangle$-tame r.e. for some $\mathfrak{A}$-r.e. set B with $B \leq_{\mathfrak{A}} A$ and there exists no $\mathfrak{A}$-r.e. set W such that $A =_{\mathfrak{A}} W$.

We will solve this problem in a way which was suggested by M. Lerman for the special case of ordinary recursion theory. It turns out that routine precautions are sufficient to make the proof work for all admissible structures.

Call a set D $\mathfrak{A}$-d.r.e. if $D = A - B$ for some $\mathfrak{A}$-r.e. sets A, B. It is easy to see that we have in this case $D \leq_{\mathfrak{A}} 0'_{\mathfrak{A}}$ if A or B is regular over L_κ.

Lemma 1. *Every $\mathfrak{A}$-d.r.e. set D is $\langle\mathfrak{A}, B\rangle$-tame r.e. for some $\mathfrak{A}$-r.e. set B with $B \leq_{\mathfrak{A}} A$.*

Proof. Assume that $D = A_1 - B_1$ with $\mathfrak{A}$-r.e. sets A_1, B_1. We can assume without loss of generality that $B_1 \subseteq A_1$. Fix an $\mathfrak{A}$-recursive 1–1 enumeration $f: \kappa \to A_1$ of the $\mathfrak{A}$-r.e. set A_1. Define $B := f^{-1}[B_1]$. It is obvious that B is $\mathfrak{A}$-r.e. We have for every $K \in L_\kappa$:

$$K \subseteq D \leftrightarrow K \subseteq A_1 - B_1 \leftrightarrow \exists \hat{K} \in L_\kappa (f[\hat{K}] = K \wedge \hat{K} \subseteq \kappa - B).$$

This shows that D is $\langle\mathfrak{A}, B\rangle$-tame r.e. Finally we have $B \leq_{\mathfrak{A}} D$ because $K \subseteq \kappa - B \leftrightarrow f[K] \subseteq D$.

In order to prove Theorem 2 it is thus enough to construct an $\mathfrak{A}$-d.r.e. set $D = A - B$ with A regular such that D is not contained in any $\mathfrak{A}$-r.e. degree. We will mainly describe the construction since the verification of the desired properties is fairly standard for this finite injury priority construction. For convenience in writing we restrict our attention to an admissible set L_α instead of an admissible structure $\mathfrak{A}$ (for which the proof is litterally the same).

D_σ, A_σ, B_σ, $W_{a,\sigma}$ are the collections of elements which have been put into these sets D, A, B, W_a before step $\sigma \in \alpha$ of the construction.

For every $e=\langle a,b,c\rangle\in\alpha$ with $a,b,c\in\alpha$ we have a requirement

$$R_e:=(\neg W_a\leqslant^b_\alpha D\vee\neg D\leqslant^c_{w\alpha}W_a).$$

(we have fixed a universal $\Sigma_1 L_\alpha$ predicate U_1 and we write W_α for $\{x\mid\langle x,a\rangle\in U_1\}$, we have further fixed an enumeration of U_1 so that the notation $W_{a,\sigma}$ makes sense).

Define $\rho:=t\sigma 2p\,\alpha$ (the tame Σ_2 projectum of α, see [4]) and fix a tame Σ_2 projection g which maps ρ 1–1 onto α. Further we fix an α-recursive tame approximation $f:\alpha\times\rho\to\alpha$ such that $f(\sigma,\cdot)$ is 1–1 for every $\sigma\in\alpha$. We further define an approximation $\{c\}^{W_{a,\sigma}}_\sigma$ for every function $\{c\}^{W_a}$ (this is the — in general partial — function which is weakly α-recursive in W_a with index c). Define

$$\{c\}^{W_{a,\sigma}}_\sigma(x)\downarrow\;:\leftrightarrow\exists\tau\leqslant\sigma\exists KHy\in L_\tau(L_\tau\models[\langle x,y,K,H\rangle\in W_c]\\ \wedge K\subseteq W_{a,\tau}\wedge H\subseteq L_\sigma-W_{a,\sigma}).$$

Observe that the σ in $W_{a,\sigma}$ at the end is not a misprint. If $\{c\}^{W_{a,\sigma}}_\sigma(x)\downarrow$ we determine the value such that $\{c\}^{W_{a,\sigma}}_\sigma(x)\simeq y$ and the negative neighborhood of this computation as follows: Choose $\tau\leqslant\sigma$ with the properties above minimal. For this τ choose $\hat y$, $\hat K$, $\hat H$ with the properties above such that $\langle\hat y,\hat K,\hat H\rangle$ is minimal with respect to the canonical $\Delta_1 L_\alpha$ well-ordering of L_α. Then we define $\{c\}^{W_{a,\sigma}}_\sigma(x)\simeq\hat y$ and $\hat H$ is defined to be the negative neighborhood of this computation.

Every requirement is at every step σ of the construction in one of the states $0,1,2$. At the beginning of the construction every requirement is in state 0.

We say that R_e requires attention at step σ if $\exists\delta<\rho(f(\sigma,\delta)=e)$ and

(1) R_e is in state 0 at the beginning of step σ and there exists some $x<\sigma$ such that

(a) $x-A_\sigma$ has an order type $\geqslant\gamma$ where $\gamma:=\sup f(\sigma,\cdot)[\delta+1]$, and

(b) $x\notin A_\sigma$, and

(c) if at some step $\sigma'<\sigma$ a requirement $R_{e'}$ received attention with $f(\sigma',\delta')=e'$ and $\delta'<\delta$ then we have $\sigma'\leqslant x$ and

(d) $\{c\}^{W_{a,\sigma}}_\sigma(x)\simeq 0$ with a negative neighborhood K (we assume that the characteristic function of a set has value 1 for arguments *in* the set), and

(e) we have for K from (d) that

$$\exists H_1H_2\in L_\sigma(L_\sigma\models[\langle K,1,H_1,H_2\rangle\in W_b]\wedge H_1\subseteq D_\sigma\wedge H_2\subseteq L_\sigma-D_\sigma)$$

(see the definition of $W_a\leqslant^b_\alpha D$); or

(2) R_e was put into state 1 at some step $\sigma'<\sigma$ and an element x was put into A at step σ' and the state of R_e was not changed after σ' and we have $\{c\}^{W_{a,\sigma}}_\sigma(x)\simeq 1$; or

(3) R_e is not in state 0 at the beginning of step σ and there exists no $\sigma'<\sigma$ and $\delta<\rho$ such that $f(\tau,\delta)\simeq e$ for all τ such that $\sigma'\leqslant\tau\leqslant\sigma$.

Construction

Step σ. If no requirement requires attention at step σ go to the next step. Otherwise choose $\delta<\rho$ minimal such that R_e with $e=f(\sigma,\delta)$ requires attention at step σ. We say then that R_e receives attention at step σ and we do the following:

At first we put all $R_{\tilde{e}}$ into state 0 such that $\tilde{e}=f(\sigma,\tilde{\delta})$ for some $\tilde{\delta}>\delta$. Then we proceed as follows

Case 1. R_e requires attention according to (1).

Choose the x in (1) minimal and put into A. Further we put R_e into state 1.

Case 2. R_e requires attention according to (2) but not according to (3).

Put the x in (2) into B and put R_e into state 2.

Case 3. R_e requires attention according to (3).

Put R_e into state 0.

End of the construction.

One proves as usual that for every $\delta<\rho$ there exists a step such that no R_e with $g^{-1}(e)\leqslant\delta$ receives attention after this step.

The condition "$x-A_\sigma$ has an order type $\geqslant\sup f(\sigma,\cdot)[\delta+1]$" in (1) (a) makes it possible to show that $\alpha-A$ has order type α and A is regular over L_α.

Assume then for a contradiction that $D=_\alpha W_a$ for some α-r.e. set W_a. It is essential that we can assume without loss of generality that W_a is regular (apply the regular set theorem). Assume that b,c are indices such that $W_a\leqslant^b_\alpha D$ and $D\leqslant^c_{w\alpha}W_a$ (it is important that we take b such that $W_a\leqslant^b_\alpha D$ — not just $W_a\leqslant^b_{w\alpha}D$). We consider then requirement R_e with $e:=\langle a,b,c\rangle$.

Consider a step σ_0 such that after step σ_0 no requirement of higher priority than R_e requires attention and such that $x\in\sigma_0-A$ exists so that $x-A$ has an order type which is large enough to satisfy the condition in (1) (a) for R_e at σ_0. A step with these properties exists by our previous remarks.

Since W_a is regular, $x\notin D$ and $D\leqslant^c_{w\alpha}W_a$ we have that $\{c\}^{W_{a,\sigma}}_\sigma(x)\simeq0$ with the same negative neighborhood K for all large enough σ and we have $K\subseteq L_\alpha-W_a$.

Since $W_a\leqslant^b_\alpha D$ there exists a step $\sigma_1\geqslant\sigma_0$ and there exist neighborhoods $H_1,H_2\in L_{\sigma_1}$ such that $H_1\subseteq D_\sigma$ for all $\sigma\geqslant\sigma_1$ and $H_2\subseteq L_\alpha-D_\sigma$ for $\sigma\geqslant\sigma_1$ (we have used for the latter that A is regular).

It follows from these observations that there is some step $\hat{\sigma}$ such that R_e is put into state 1 at step $\hat{\sigma}$, some $\hat{x}$ is put into A at step $\hat{\sigma}$ (call the associated negative neighborhood $\hat{K}$, thus $\hat{K}\subseteq L_{\hat{\sigma}}-W_{a,\hat{\sigma}}$) and R_e is not put into state 0 at any step after $\hat{\sigma}$.

Then either R_e is in state 1 at all steps after $\hat{\sigma}$ — in which case we have $\hat{x}\in A-B=:D$ but not $\{c\}^{W_a}(x)\simeq1$, a contradiction.

Or there exists a first step $\check{\sigma}>\hat{\sigma}$ such that R_e is put into state 2 at step $\check{\sigma}$, $\hat{x}$ is put into B at step $\check{\sigma}$ and R_e is in state 2 at all steps after $\check{\sigma}$. In this case we have that some element $y\in\hat{K}$ was enumerated into W_a at some step between $\hat{\sigma}$ and $\check{\sigma}$ although some computation $\langle\hat{K},1,\hat{H}_1,\hat{H}_2\rangle\in W_b$ with $\hat{H}_1\subseteq D_{\hat{\sigma}}\wedge H_2\subseteq L_{\hat{\sigma}}-D_{\hat{\sigma}}$ existed at step $\hat{\sigma}$. This computation may have been injured (if $\hat{x}\in\hat{H}_2$) at the end of step $\hat{\sigma}$ by putting $\hat{x}$ into A, but this $\hat{x}$ is the only possible injury of the computation. Since one puts $\hat{x}$ into $B\subseteq L_\alpha-D$ at step $\check{\sigma}$ the computation is restored in any case at step $\check{\sigma}$. Further this computation in D will remain valid at all steps after $\check{\sigma}$ so that we have in fact $\hat{H}_1 \subseteq D \wedge \hat{H}_2 \subseteq L_\alpha - D$. This contradicts our assumption $W_a \leq^b_\alpha D$ since we had observed before that $\neg\hat{K} \subseteq L_\alpha - W_a$.

This finishes the proof of Theorem 2.

2. α-recursively enumerable degrees

Not much is proved so far about the jump of α-r.e. degrees if α is Σ_2 admissible or $\alpha > \sigma 2 \operatorname{cf} \alpha \geq \sigma 2 p\, \alpha$ (these are the types (1) and (2) in the classification of Section 3 in [6]), whereas everything is known about the jump of α-r.e. degrees for the other α (see [6]). We will study in this chapter mainly type (2). The result for type (1) is contained in Theorem 5.

For α of type (2) we know so far that there exist nonzero α-r.e. degrees which are low and incomplete α-r.e. degrees which are high. This agrees with the situation in ordinary recursion theory. On the other hand we learnt in the meantime that the structure of $\Sigma_2 L_\alpha$ degrees above $0'$ is for this type very different from the corresponding structure in ordinary recursion theory: There exists the distinguished degree $0^{\frac{3}{2}}$ between $0'$ and $0''$ which was described in Lemma 7, Section 2 in [6]. Observe that for α of type (2) there exist always incomplete non-hyperregular α-r.e. degrees so that the assumption of this Lemma is satisfied. We can apply the results about weakly inadmissible structures from [5] and the preceding Section 1. The structure $\mathfrak{B} := \langle L_\alpha, C\rangle$ where $C \in 0'$ is α-r.e. and regular is weakly inadmissible if α is of type (2) and it is obvious that the $\mathfrak{B}$-r.e. degrees are isomorphic to the $\Sigma_2 L_\alpha$ degrees above $0'$ in L_α.

It is easy to see that $0^{\frac{3}{2}}$ is the boundary between the jumps of hyperregular α-r.e. degrees and the jumps of non-hyperregular α-r.e. degrees for α of type (2):

If $\boldsymbol{a}$ is a hyperregular α-r.e. degree, then $\boldsymbol{a}'$ is a tame $\Sigma_2 L_\alpha$ degree and therefore we have $\boldsymbol{a}' \leq_\alpha 0^{\frac{3}{2}}$ because $0^{\frac{3}{2}}$ is the greatest tame $\Sigma_2 L_\alpha$ degree.

If $\boldsymbol{a}$ is a non-hyperregular α-r.e. degree then we have that the complete $\Sigma_2 L_\alpha$ set U_2 is weakly α-recursive in $\boldsymbol{a}'$ (Shore [11]). According to Lemma 7 in [6] we have therefore $0^{\frac{3}{2}} \leq_\alpha \boldsymbol{a}'$.

In particular we have thus shown that $0^{\frac{3}{2}}$ is comparable with the jump of every α-r.e. degree. But we do not yet know so far whether there exist for α of type (2) any other jumps of α-r.e. degrees besides $0'$ and $0''$.

Theorem 3. *Assume α is admissible and $\alpha > \sigma 2 \operatorname{cf} \alpha \geq \sigma 2 p\, \alpha$. Then there exists a non-hyperregular α-r.e. degree $\boldsymbol{a}$ such that $\boldsymbol{a}' = 0^{\frac{3}{2}}$.*

Proof. One runs into a lot of trouble if one tries to prove this Theorem as one would do it in ordinary recursion theory, i.e. if one fixes a set $S \in 0^{\frac{3}{2}}$ and tries to construct A as a suitable "thick subset" as in Soare [14] in order to get $A' =_\alpha 0^{\frac{3}{2}}$.

Therefore we prove Theorem 3 as follows: We make sure that the order type of $\alpha - A$ is less than α so that A is non-hyperregular as in Shore [9]. This implies that $0^{\frac{3}{2}} \leq_\alpha A'$.

We keep the jump down in order to get $0^{\frac{3}{2}} =_\alpha A'$ by following the usual strategy which is applied to make a constructed set low (see e.g. Soare [14] Theorem 4.1.). This strategy will make sure that "$K \in L_\alpha \wedge K \subseteq A'$" is $\Pi_2 L_\alpha$ which implies $A' \leq_\alpha 0^{\frac{3}{2}}$ by the special properties of $0^{\frac{3}{2}}$.

For the exact proof we fix a tame $\Sigma_2 L_\alpha$ projection P from L_α onto $t\sigma 2p\,\alpha = \sigma 2 \text{ cf } \alpha =: \kappa$ and an α-recursive tame approximation $P(\cdot,\cdot): \alpha \times L_\alpha \to \alpha$ such that $P(\sigma, \cdot)$ is 1–1 for every $\sigma \in \alpha$.

Further we fix a cofinal strictly increasing and continuous $\Sigma_2 L_\alpha$ function $g: \kappa \to \alpha$ and an α-recursive approximation $g(\cdot,\cdot): \alpha \times \kappa \to \alpha$ such that $g(\sigma, \cdot) \leq \sigma$ and $g(\sigma, \cdot)$ is weakly increasing. According to the definition of the jump in α-recursion theory (see Shore [11]) there exists an α-r.e. set W such that for every $M \subseteq L_\alpha$:

$$M' = \{y \mid \exists H_1 H_2(\langle y, H_1, H_2\rangle \in W \wedge H_1 \subseteq M \wedge H_2 \subseteq L_\alpha - M)\}.$$

We fix an α-recursive enumeration of W such that $W_\sigma \subseteq L_\sigma$ for every σ.

We define a restriction function $r(i, \sigma)$ for arguments $i \in \kappa$ and $\sigma \in \alpha$. $r(i, \sigma)$ will be the α-finite set of those elements less than σ which are kept out from A at step σ with priority i.

Fix i and σ for the following definition of $r(i, \sigma)$. Let K be the set of those elements $x \in \sigma - A_\sigma$ such that $x - A_\sigma$ has an order type less than i. Further we check for every $j < i$ whether the following condition $(*j)$ is satisfied:

$$\begin{aligned}\exists \tau \leq \sigma \exists y (\forall \tau' (\tau \leq \tau' \leq \sigma \to P(\tau', y) = j) \wedge \exists H_1 H_2 (\langle y, H_1, H_2\rangle \\ \in W_\tau \wedge H_1 \subseteq A_\tau \wedge L_\tau \models [\text{card}\,(H_2) < \kappa] \wedge H_2 \subseteq L_\tau - A_\sigma)).\end{aligned}$$

If $(*j)$ is satisfied we choose τ in this condition minimal. For this τ let $\langle y, \hat{H}_1, \hat{H}_2\rangle$ be the minimal tripel (with respect to $<_{L_\alpha}$) which satisfies $(*j)$. We define then $K_j := \hat{H}_2$. If $(*j)$ is not satisfied we define $K_j := \emptyset$. Then we define

$$r(i, \sigma) := K \cup \bigcup \{K_j \mid j < i\}.$$

Since κ is a regular α-cardinal we have $\alpha\text{-card}\,(r(i, \sigma)) < \kappa$ for every i, σ.

We have positive requirements P_i for every $i < \kappa$ which try to make sure that the order type of $g(i) - A$ is less than κ.

Construction. Step σ. Choose $i < \kappa$ minimal such that

$$g(\sigma, i) - A_\sigma - r(i, \sigma) \neq \emptyset.$$

If such an i does not exist go to the next step. Otherwise for this i we say that P_i receives attention at σ. We put all elements of $g(\sigma, i) - A_\sigma - r(i, \sigma)$ into A.

End of the construction.

Claim 1. For all $i < \kappa$ there exists a step σ_i such that no P_j with $j \leq i$ receives attention after step σ_i.

Proof. Assume i_0 is minimal such that σ_{i_0} does not exist. Since $i_0 < \sigma 2 \text{ cf } \alpha$ there exists a step σ' such that no P_j with $j < i_0$ receives attention after σ'.

There exists $\hat{\sigma} \geqslant \sigma'$ such that

$$\forall \sigma \geqslant \hat{\sigma}(\forall i \leqslant i_0(g(\sigma, i) = g(i) < \sigma) \wedge \forall y \in L_\alpha((P(y) \leqslant i_0 \\ \rightarrow P(\sigma, y) = P(y)) \wedge (P(y) > i_0 \rightarrow P(\sigma, y) > P(y))).$$

Choose $\tau_1 > \hat{\sigma}$ minimal such that P_{i_0} receives attention at step τ_1. Then all elements of $g(i_0) - A_{\tau_1} - r(i_0, \tau_1)$ are put into A at step τ_1.

Choose $\tau_2 > \tau_1$ minimal such that P_{i_0} receives attention at τ_2. This implies that $g(i_0) - A_{\tau_2} - r(i_0, \tau_2) \neq \emptyset$. That is only possible if there exists some $y \in r(i_0, \tau_1)$ such that $y \notin r(i_0, \tau_2) \wedge y \notin A_{\tau_2}$. Then there exists some $j < i_0$ such that $y \in K_j$ where K_j is some negative neighborhood $\hat{H}_2 \subseteq L_{\tau_1} - A_{\tau_1}$ (see the definition of $r(i_0, \tau_1)$) and such that a minimal step τ with $\tau_1 < \tau < \tau_2$ exists at which some $z \in \hat{H}_2$ is put into A. By the choice of τ_1, τ_2 some requirement P_i with $i > i_0$ must then receive attention at τ. But since $\hat{H}_2 \subseteq \tau - A_\tau$ one has for this i, τ that $\hat{H}_2 \subseteq r(i, \tau)$ so that no element of $\hat{H}_2$ is put into A at step τ. Contradiction.

Claim 2. $\alpha - A$ is unbounded in α and has order type κ. Further A is regular and non-hyperregular.

Proof. (a) Assume for a contradiction that some γ less than κ is the order type of $\alpha - A$.

Go to a stage σ such that after σ no P_i with $i \leqslant \gamma$ receives attention. Consider the minimal j such that requirement P_j receives attention at some step $\tau > \sigma$. We have then $g(\tau, j) - A_\tau - r(j, \tau) \neq \emptyset$ and therefore $\tau - A_\tau$ has an order type $\geqslant j$ by the definition of $r(j, \tau)$. The first j elements of $\tau - A_\tau$ will never be put into A by the choice of $\tau > \sigma$. This is a contradiction to $j > \gamma$.

(b) Assume for a contradiction that some $\delta < \alpha$ exists such that $\delta - A$ has order type κ.

Choose i such that $g(i) > \delta$ and σ such that no P_j with $j \leqslant i$ receives attention at σ and $g(\sigma, i) = g(i)$.

Then we have $g(\sigma, i) - A_\sigma - r(i, \sigma) \neq \emptyset$ because $r(i, \sigma)$ has α-cardinality less than κ and $g(\sigma, i) - A$ has order type κ. Therefore some P_j with $j \leqslant i$ receives attention at σ which is a contradiction to the choice of σ.

A is regular because $\delta - A$ has an order type less than $\kappa = 2\,\mathrm{cf}\,\alpha$ for every $\delta < \alpha$. A is non-hyperregular because the function $f: \kappa \to \alpha$, where $f(i)$ is the i-th element of $\alpha - A$, is cofinal and weakly α-recursive in A.

Claim 3. $A' =_\alpha 0^{\frac{3}{2}}$.

Proof. We have

$$K \subseteq A' \leftrightarrow \forall i < \kappa \forall \sigma\, \forall y((y \in K \wedge P(y) = i \wedge \forall \tau \geqslant \sigma((\text{no } P_j \\ \text{with } j \leqslant i \text{ receives attention at } \tau) \wedge P(\tau, y) = i)) \\ \rightarrow \exists \tau \geqslant \sigma \exists H_1 H_2(\langle y, H_1, H_2 \rangle \in W_\tau \wedge H_1 \subseteq A_\tau \wedge H_2 \\ \subseteq \tau - A_\tau \wedge L_\tau \models [\mathrm{card}\,(H_2) < \kappa])).$$

In order to prove this equivalence we use for "$\rightarrow$" that A is regular and that $\delta - A$ has an order type less than κ for every $\delta < \alpha$. For "$\leftarrow$" we consider some $y \in K$. Then there exist $i = P(y)$ and σ such that the premise of the right side is satisfied. Therefore there exists $\tau \geq \sigma$ as in the conclusion on the right side. Thus there exists some tripel $\langle y, \hat{H}_1, \hat{H}_2 \rangle \in W$ such that $\hat{H}_1 \subseteq A_\tau$, $\hat{H}_2 \subseteq \tau - A_\tau$ and $\hat{H}_2 \subseteq r(j, \tau')$ for every $\tau' \geq \tau$ and $j > i$. Therefore no element of $\hat{H}_2$ will be put into A at any step $\tau' \geq \tau$ because no P_j with $j \leq i$ receives attention after $\tau \geq \sigma$. This shows that $y \in A'$.

The right side of the equivalence above can obviously be written in $\Pi_2 L_\alpha$ form so that we have

$$K \in L_\alpha \wedge K \subseteq A' \leftrightarrow \langle K, e \rangle \notin U_2$$

for some fixed index e where U_2 is a universal $\Sigma_2 L_\alpha$ predicate. Since we have $U_2 \leq_{w\alpha} 0^{\frac{3}{2}}$ by Lemma 7 in [6] we have thus expressed $K \subseteq A'$ α-recursively in $0^{\frac{3}{2}}$.

Since $K \in L_\alpha \wedge K \subseteq L_\alpha - A'$ is as well $\Pi_2 L_\alpha$ (trivial) we have shown that $A' \leq_\alpha 0^{\frac{3}{2}}$.

Finally $0^{\frac{3}{2}} \leq_\alpha A'$ follows from the non-hyperregularity of A.

This finishes the proof of Theorem 3.

Theorem 4. *Assume α is admissible and $\alpha > \sigma 2\mathrm{cf}\, \alpha \geq \sigma 2p\, \alpha$. If D is a tame $\Sigma_2 L_\alpha$ set with $0' \leq_\alpha D$, then there exists a hyperregular α-r.e. set A such that $A' =_\alpha D$.*

Proof. The general strategy will be that one which is used in ordinary recursion theory in order to prove Sack's jump theorem [7] (see Soare [14]). This strategy is a variation of the strategy which is used in order to construct incomplete high r.e. sets. The positive requirements are the same but instead of using Sack's preservation strategy one tries here to keep the jump down by preserving computations which predict that some element is going to be in the jump of the constructed set. This is the same preservation strategy as in the construction of non-zero low degrees and we have used this strategy already in the proof of the preceding Theorem.

If one wants to transfer this construction from ordinary recursion theory to α-recursion theory one has to overcome similar problems as in the construction of incomplete high α-r.e. degrees (Theorem 1 in [6]). But most of these problems have to be solved here in a different way because of slight differences in the situation.

We have again the difficulty that the construction from ordinary recursion theory gives only $D \leq_{w\alpha} A'$ instead of $D \leq_\alpha A'$. Here we have to keep the constructed set A hyperregular and therefore we use the regular set theorem from β-recursion theory. According to Theorem 4 in [5] there exists a tame $\Sigma_2 L_\alpha$ set $\hat{D}$ in the α-degree of D which is regular and satisfies in addition for every set $B \subseteq L_\alpha : D \leq_{w\alpha} B \leftrightarrow D \leq_\alpha B$. Observe that this escape was not possible in the construction of incomplete high α-r.e. degrees because $0''$ does not contain a set with the latter property if α is of type (2).

In Theorem 1 in [6] we could get along without a regular representative in $0''$ because we made the priority list extremely short. For the present proof one needs a better approximation to the priority list than there because here we have the additional requirement to keep the jump down. Thus it is a lucky circumstance that due to the regular set theorem from β-recursion theory we can work here with a regular representative and use an — in general longer — priority list of length $\sigma 2 \text{ cf } \alpha = t\sigma 2p\, \alpha$.

At limit points of the priority list we have here the same problem with the inductive argument as in [6]. This problem was described there in point 4) of the motivation before Theorem 1. Similar as there we use a fixpoint argument in order to get along although the induction hypothesis is too weak at limit points.

For the exact proof consider the weakly inadmissible structure $\mathfrak{B} := \langle L_\alpha, C\rangle$ with $C \in 0'$ α-r.e. and regular. Since D is $\mathfrak{B}$-tame r.e. there exists by Theorem 1 in [5] a regular $\mathfrak{B}$-tame r.e. set $S \subseteq \alpha$ such that

$$\forall B \subseteq L_\alpha (S \leqslant_{\mathrm{w}\mathfrak{B}} B \leftrightarrow S \leqslant_{\mathfrak{B}} B), \qquad D =_{\mathfrak{B}} S$$

and every $K \subseteq S$ with $K \in L_\alpha$ has an order type less than $\sigma 2 \text{ cf } \alpha$ (we need the latter property for the proof of Claim 1).

We define $\hat{D} := C \vee S$ and have then $D =_\alpha \hat{D}$, $\hat{D}$ is regular and tame $\Sigma_2 L_\alpha$ and $\hat{D} \leqslant_{\mathrm{w}\alpha} B \leftrightarrow \hat{D} \leqslant_\alpha B$ for all $B \subseteq L_\alpha$ such that $0' \leqslant_\alpha B$.

Fix Δ_0-formulas ϕ, $\hat{\psi}$ such that

$$x \in C \leftrightarrow L_\alpha \models \exists y \phi(x, y)$$

and

$$x \in S \leftrightarrow L_\alpha \models \exists y\, \forall z ((y \text{ is an ordinal}) \wedge \hat{\psi}(x, y, z)).$$

Define

$$\psi(x, y, z) :\equiv (y \text{ is an ordinal}) \wedge \hat{\psi}(x, y, z).$$

We will construct A as a "thick subset" of the α-r.e. set $R \subseteq \alpha$ which is defined as follows:

$$\langle u, v\rangle \in R \leftrightarrow u, v \in \alpha \wedge (\exists x (v = 2x \wedge \neg L_u \models \exists y \phi(x, y)) \\ \vee \exists x (v = 2x + 1 \wedge \forall y \leqslant u \exists z \neg \psi(x, y, z))).$$

We have

$$v \notin \hat{D} \leftrightarrow \{u \mid \langle u, v\rangle \in R\} = \alpha,$$

$$v = 2x \wedge x \in C \leftrightarrow \{u \mid \langle u, v\rangle \in R\} = \mu\gamma(L_\gamma \models \exists y \phi(x, y)) \in \alpha$$

and

$$v = 2x + 1 \wedge x \in S \leftrightarrow \{u \mid \langle u, v\rangle \in R\} = \mu\gamma(\forall z \psi(x, \gamma, z)) \in \alpha.$$

We write in the following for any set $M \subseteq \alpha$:

$$M^{(v_0)} := \{\langle u, v\rangle \in M \mid v = v_0\}$$

and

$$M^K := \bigcup\{M^{(v)} \mid v \in K\} \quad \text{for any set } K.$$

Claim 1. For every set $\hat{K} \in L_\alpha$ there exist $K, H \in L_\alpha$ such that $R^{\hat{K}} = H \cup \alpha \times K$. In particular $R^{\hat{K}}$ is α-recursive.

Proof. Define $K_1, K_2 \in L_\alpha$ by $\hat{D} \cap \hat{K} = K_1 \vee K_2$ (we use here the regularity of $\hat{D}$).

Define a function $f: K_1 \to \alpha$ such that $f(x) = \mu\gamma(L_\gamma \models \exists y \phi(x, y))$. Then f is α-finite because f is α-recursive and $\operatorname{dom} f = K_1 \in L_\alpha$. Therefore R^{2K_1} is some α-finite set H_1 (we had defined $2K_1 := \{2x \mid x \in K_1\}$).

Further by the choice of $\hat{D}$ respectively S we have that $\alpha\text{-card}(K_2) < \sigma 2 \operatorname{cf} \alpha$. Since $R^{(v)}$ is α-finite for every $v \in K_2$ we get that R^{2K_2+1} is some α-finite set H_2.

We have then $R^{\hat{K}} := H \cup \alpha \times K$ with the α-finite sets $H := H_1 \cup H_2$, $K := \hat{K} - \hat{D} \cap \hat{K}$. $\square$

Fix as in the proof of Theorem 3 an α-r.e. set W such that

$$M' = \{y \mid \exists H_1 H_2 \in L_\alpha(\langle y, H_1, H_2\rangle \in W \wedge H_1 \subseteq M \wedge H_2 \subseteq \alpha - M\}.$$

We assume here for trivial technical reasons that $M' \subseteq \alpha$ (use some α-recursive function which maps L_α 1–1 onto α).

We fix α-recursive enumerations of the sets W and R such that $W_\sigma \subseteq L_\sigma$ and $R_\sigma \subseteq L_\sigma$ for all $\sigma \in \alpha$.

As usual we write A_σ, W_σ etc. for the set of those elements which are enumerated *before* step σ. Further we write $R_\sigma^{(v)}$, R_σ^K instead of $R^{(v)} \cap R_\sigma$ respectively $R^K \cap R_\sigma$.

The restriction function r will be defined in two parts. First we define a restriction function q which is needed in order to keep the jump A' down. Then we define a restriction function $\hat{r}$ which is needed in order to make A hyperregular (we use the standard strategy in order to make A hyperregular).

For $\gamma, \sigma \in \alpha$ we define $q(\gamma, \sigma)$ as follows: Check whether some step $\tau < \sigma$ exists such that

$$\exists H_1 H_2(\langle \gamma, H_1, H_2\rangle \in W_\tau \wedge H_1 \subseteq A_\tau \wedge H_2 \subseteq \tau - A_\sigma).$$

If τ does not exist define $q(\gamma, \sigma) := 0$. Otherwise we take the least such τ and we take for this τ the least $z \in \alpha$ such that the existing sets H_1, H_2 can chosen to be subsets of z. We define then

$$q(\gamma, \sigma) := \max\{z, 1\}.$$

In order to define $\hat{r}$ we first have to define analogously as in Soare [14] the two functions $l:\alpha\times\alpha\to\alpha$ and $u:\alpha\times\alpha\times\alpha\to\alpha$. Define

$$l(e,\sigma):=\begin{cases}\min(\{x\mid\{(e)_0\}^{A_\sigma}_\sigma(x)\uparrow\}\cup\{\alpha^*,(e)_1\}) & \text{if}\quad \alpha^*<\alpha,\\ \min(\{x\mid\{(e)_0\}^{A_\sigma}_\sigma(x)\uparrow\}\cup\{(e)_1\}) & \text{if}\quad \alpha^*=\alpha.\end{cases}$$

We use in this definition the two projection functions $(\cdot)_0$ and $(\cdot)_1$ which are associated with the pairing function $\langle\cdot,\cdot\rangle:\alpha\times\alpha\to\alpha$. The approximations $\{e\}^{A_\sigma}_\sigma(x)$ to the functions which are weakly recursive in A are defined as follows: We write $\{e\}^{A_\sigma}_\sigma(x)\downarrow$ if

$$\exists\tau<\sigma\exists yH_1H_2(L_\tau\models[\langle x,y,H_1,H_2\rangle\in W_e]\wedge H_1\subseteq A_\tau\wedge H_2\subseteq L_\tau-A_\sigma).$$

The "use function" u is defined as follows: We set $u(e,x,\sigma)=0$ if $\{e\}^{A_\sigma}_\sigma(x)\uparrow$. If $\{e\}^{A_\sigma}_\sigma(x)\downarrow$ we go back to the definition of $\{e\}^{A_\sigma}_\sigma(x)\downarrow$ and choose the existing $\tau<\sigma$ minimal. For this τ we take the minimal tripel $\langle\hat{y},\hat{H}_1,\hat{H}_2\rangle$ which satisfies the condition in the definition. We define then $u(e,x,\sigma)$ as the minimal $z\in\alpha$ such that $\hat{H}_2\subseteq z$. The $\hat{y}$ out of this minimal tripel is defined to be the value of $\{e\}^{A_\sigma}_\sigma(x)$ and we write then $\{e\}^{A_\sigma}_\sigma(x)\simeq\hat{y}$.

Finally we define

$$\hat{r}(e,\sigma):=\sup\{u((e)_0,x,\sigma)\mid x<l(e,\sigma)\}$$

and

$$r(e,\sigma):=\max(\hat{r}(e,\sigma),q(e,\sigma)).$$

Observe that we always have $r(e,\sigma)\leq\sigma$.

In order to assign priorities we fix a tame Σ_2L_α projection P which maps α 1–1 onto $\kappa:=\sigma2\operatorname{cf}\alpha$. We fix an α-recursive tame approximation $P(\cdot,\cdot):\alpha\times\alpha\to\alpha$ such that $P(\sigma,\cdot)$ is 1–1 for every $\sigma\in\alpha$ and

$$\forall\gamma<\kappa\exists\sigma\forall\sigma'\geq\sigma\forall z((P(z)\leq\gamma\to P(\sigma',z)=P(z))\\ \wedge(P(z)>\gamma\to P(\sigma',z)>\gamma)).$$

Construction. Step σ. Consider every $x=\langle x',e\rangle\in R_{\sigma+1}$ which is not already element of A_σ. We put x into A at step σ if and only if $x\geq r(i,\sigma)$ for every $i\leq\sigma$ such that $P(\sigma,i)\leq P(\sigma,e)$.

End of the construction.

Claim 2. Consider an index e and a step σ_0 such that

$$\forall\sigma\geq\sigma_0\forall z((P(z)\leq P(e)\to(P(\sigma,z)=P(z)\wedge z<\sigma_0))\\ \wedge(P(z)>P(e)\to P(\sigma,z)>P(e))).$$

Define $K_e:=\{x\mid P(x)<P(e)\}$. Then the following holds: If $\sigma>\sigma_0$ is a step such that $R^{K_e}\cap\sigma=R^{K_e}_\sigma\cap\sigma$ and no element of $R^{K_e}\cap\sigma$ is put into A at step σ then no $x<r(e,\sigma)$ is put into A at any stage $\sigma'\geq\sigma$ and we have $\forall\sigma'\geq\sigma(r(e,\sigma')\geq r(e,\sigma))$.

Proof. Induction on $P(e)$. Assume for a contradiction that there exists a minimal $\sigma'\geq\sigma$ such that some $x<r(e,\sigma)$ is put into A at step σ'.

Then we have $\sigma' > \sigma$ because for $\sigma' = \sigma$ the ordinal $x < r(e, \sigma)$ would be an element of $R^{K_e} \cap \sigma$. The computations in A which contribute to the definition of $r(e, \sigma)$ are not destroyed before σ' because of the minimality of σ'. Therefore we have $r(e, \sigma') \geq r(e, \sigma)$ so that the element $x = \langle x', e' \rangle < r(e, \sigma)$ which is put into A at step σ' is an element of $R^{K_e} \cap \sigma$ and we have $P(e') < P(e)$. Further since x is not put into A at step σ there exists some $\hat{e}$ with $P(\hat{e}) \leq P(e') < P(e)$ such that $x < r(\hat{e}, \sigma)$. But this situation is impossible according to the induction hypothesis for $P(\hat{e})$. Contradiction.

It remains to show that $r(e, \sigma') \geq r(e, \sigma)$ for $\sigma' \geq \sigma$ but this follows immediately from the preceding because the computations in A which contribute to the definition of $r(e, \sigma)$ are never destroyed.

Claim 3. For every $e \in \alpha$ we have $A^{K_e} =^* R^{K_e}$ where $K_e := \{i \mid P(i) < P(e)\}$ and $M_1 =^* M_2 :\leftrightarrow M_1 - M_2 \in L_\alpha \wedge M_2 - M_1 \in L_\alpha$.

Proof. Induction on $P(e)$. We define for the proof

$$R(e, \sigma) := \sup \{r(i, \sigma) \mid i \leq \sigma \wedge P(\sigma, i) \leq P(\sigma, e)\}.$$

Further we fix a step σ_0 such that

$$\forall \sigma \geq \sigma_0 \, \forall z ((P(z) \leq P(e') \rightarrow (P(\sigma, z) = P(z) \wedge z < \sigma_0)) \wedge (P(z) > P(e') \rightarrow P(\sigma, z) > P(e'))).$$

Case 1. $P(e') = P(e) + 1$ for some e'.

We show that there exists a step σ_1 and a constant r_1 such that

$$\forall \sigma \geq \sigma_1 (R(e, \sigma) \geq r_1 \wedge \exists \tau \geq \sigma (R(e, \tau) = r_1)).$$

By the induction hypothesis we have $A^{K_e} =^* R^{K_e}$. Therefore A^{K_e} is α-recursive (use Claim 1)) and regular. Thus the set

$$M_e := \{\sigma \geq \sigma_0 \mid A^{K_e} \cap \sigma = A_\sigma^{K_e} \cap \sigma\}$$

of fixpoints is unbounded in α and α-recursive.

A computation in A which contributes to the definition of $R(e, \sigma)$ for some $\sigma \geq \sigma_0$ can only be destroyed later by some element of A^{K_e} which is enumerated into A. This implies that for $\sigma \in M_e$ the computations in A which contribute to the definition of $R(e, \sigma)$ will never be destroyed.

Define for $i \in K_{e'}$ $l(i) := \sup \{l(i, \sigma) \mid \sigma \in M_e\}$ and

$$H_i := \begin{cases} \{x+1 \mid x < l(i)\} & \text{if } \forall \sigma \geq \sigma_0 (\sigma \in M_e \rightarrow q(i, \sigma) = 0), \\ \{0\} \cup \{x+1 \mid x < l(i)\} & \text{otherwise.} \end{cases}$$

Then $K := \bigcup \{H_i \times \{i\} \mid i \in K_{e'}\}$ is α-finite because K is α-r.e., α-card $(K_{e'}) < \sigma 2 \operatorname{cf} \alpha$ and every H_i is α-finite.

We define an α-recursive function $f: K \to \alpha$ such that

$$f(x) = \begin{cases} \mu\sigma \in M_e(q(i, \sigma) \neq 0) & \text{if} \quad x = \langle 0, i \rangle, \\ \mu\sigma \in M_e(l(i, \sigma) > x') & \text{if} \quad x = \langle x'+1, i \rangle. \end{cases}$$

f is in fact α-finite because $\operatorname{dom} f = K$ is α-finite.

Choose $\sigma_1 \in M_e$ such that $\operatorname{Rg} f \subseteq \sigma_1$. Define $r_1 := R(e, \sigma_1)$. Then σ_1, r_1 have the desired properties.

Choose $r_2 \geqslant r_1, \sigma_1$ such that $R^{(e)} \cap r_1 = R^{(e)}_{r_2} \cap r_1$ and $r_2 \in M_e$. Then we have $A^{(e)} \cap r_1 = A^{(e)}_{r_2+1} \cap r_1 \in L_\alpha$ and $A^{(e)} - r_1 = R^{(e)} - r_1$ by the properties of σ_1, r_1.

Case 2. $P(e')$ is a limit ordinal.

In this case we have $\omega < \sigma 2 \operatorname{cf} \alpha$. Therefore the set

$$M_{e'} := \{\sigma > \sigma_0 \mid R^{K_{e'}} \cap \sigma = R^{K_{e'}}_\sigma \cap \sigma \wedge A^{K_{e'}} \cap \sigma = A^{K_{e'}}_\sigma \cap \sigma\}$$

is unbounded in α.

Further consider the set

$$N_{e'} := \{\sigma > \sigma_0 \mid R^{K_{e'}} \cap \sigma = R^{K_{e'}}_\sigma \cap \sigma \wedge (\text{no element } x \in R^{K_{e'}} \cap \sigma$$
$$\text{is put into } A \text{ at step } \sigma)\}.$$

$N_{e'}$ is α-recursive by Claim 1 and by using Claim 2 we get that $M_{e'} = N_{e'}$. Therefore $M_{e'}$ is α-recursive which implies that $A^{K_{e'}}$ is α-recursive as well.

Thus the set $H := R^{K_{e'}} - A^{K_{e'}}$ is α-r.e. Since by the induction hypothesis every $H^{(i)}$ is α-finite for $i \in K_{e'}$ and since α-card$(K_{e'}) < \sigma 2 \operatorname{cf} \alpha$ we get that H is in fact α-finite.

Claim 4. A is hyperregular.

Proof. Assume for a contradiction that $\rho = \operatorname{rcf} A < \alpha$. We have then $\rho \leqslant \alpha^*$ since α^* is the greatest α-cardinal if $\alpha^* < \alpha$.

There is an index $i \in \alpha$ such that $\{i\}^A$ is a cofinal function from ρ into α and such that

$$\forall \gamma < \rho \; \exists \sigma' \; \forall \sigma \geqslant \sigma' \; \forall x \leqslant \gamma (\{i\}^{A_\sigma}_\sigma(x)\downarrow).$$

We consider then $e := \langle i, \rho \rangle$ and the α-finite set

$$K_e := \{y \mid P(y) < P(e)\}.$$

Fix a step σ_0 such that

$$\forall \sigma \geqslant \sigma_0 \; \forall z ((P(z) \leqslant P(e) \to (P(\sigma, z) = P(z) \wedge z < \sigma_0))$$
$$\wedge (P(z) > P(e) \to P(\sigma, z) > P(e))).$$

A^{K_e} is α-recursive (and therefore regular) according to Claim 3. Therefore the set

$$M_e := \{\sigma > \sigma_0 \mid A^{K_e} \cap \sigma = A^{K_e}_\sigma \cap \sigma\}$$

is unbounded in α and α-recursive.

We have then for every $x \in \rho$ and $y \in \alpha$:

$$\{i\}^A(x) \simeq y \leftrightarrow \exists \sigma \in M_e(1(e,\sigma) > x \wedge \{i\}_\sigma^{A_\sigma}(x) \simeq y).$$

Thus we have found an α-recursive definition of the cofinal function $\{i\}^A$ which contradicts the admissibility of α.

Claim 5. $A' \leqslant_\alpha D$.

Proof. A' is tame $\Sigma_2 L_\alpha$ since A is hyperregular and α-r.e. Therefore "$K \in L_\alpha \wedge K \subseteq A'$" is a $\Sigma_2 L_\alpha$ fact and since $0' \leqslant_\alpha D$ we can express this fact α-recursively in D.

For the other part of the reduction we observe that "$K \subseteq L_\alpha - A'$" is a $\Pi_1\langle L_\alpha, A\rangle$ fact. Therefore it is enough to express $e \notin A'$ α-recursively in D. We have

$$\begin{aligned} e \notin A' \leftrightarrow \exists p\gamma\sigma_0(&\gamma = P(e) \wedge p = P(\sigma_0, \cdot)^{-1} \restriction (\gamma+1)\\ &\wedge \operatorname{dom} p = (\gamma+1) \wedge \forall xy\sigma((\sigma \geqslant \sigma_0 \wedge \langle x, y\rangle \in p) \to P(\sigma, y) = x)\\ &\wedge \exists K_e K H \in L_\alpha(K_e = \operatorname{Rg}(p \restriction \gamma) \wedge R^{K_e} = H \cup \alpha \times K\\ &\wedge \forall \sigma \geqslant \sigma_0((R_\sigma^{K_e} \cap \sigma = (H \cup \alpha \times K) \cap \sigma \wedge (\text{no element of } R_\sigma^{K_e} \cap \sigma\\ &\text{is put into } A \text{ at step } \sigma)) \to q(e,\sigma) = 0))).\end{aligned}$$

The existence of the α-finite sets K, H on the right side was shown in Claim 1 by using the regularity of $\hat{D}$. It is easy to see that one can express "$R^{K_e} = H \cup \alpha \times K$" α-recursively in D. The α-finite function p on the right side is only mentioned in order to be able to describe the properties of σ_0. σ_0 plays a similar role as in the preceding claims.

The proof of the equivalence is then immediate from the following observation: Take $\gamma := P(e)$ and let σ_0 be a step such that

$$\forall \sigma \geqslant \sigma_0 \forall z \leqslant \gamma(P(\sigma, \cdot)^{-1}(z)\downarrow \wedge P(\sigma, \cdot)^{-1}(z) = P^{-1}(z)).$$

Define $K_e := \{y \mid P(y) < P(e)\}$. Define

$$\begin{aligned} M_e := \{\sigma \geqslant \sigma_0 \mid R_\sigma^{K_e} \cap \sigma = R^{K_e} \cap \sigma \wedge (&\text{no element of } R^{K_e} \cap \sigma \text{ is put}\\ &\text{into } A \text{ at step } \sigma)\}.\end{aligned}$$

Since R^{K_e} and A^{K_e} are α-recursive according to Claim 3 there exists an unbounded set of steps σ where

$$R_\sigma^{K_e} \cap \sigma = R^{K_e} \cap \sigma \wedge A_\sigma^{K_e} \cap \sigma = A^{K_e} \cap \sigma.$$

Therefore the set M_e is unbounded in α which implies that

$$\forall \sigma \in M_e(q(e,\sigma) = 0) \to e \notin A'$$

(use the regularity of A). Further Claim 2 implies that

$$\exists \sigma \in M_e(q(e,\sigma) > 0) \to e \in A'.$$

Thus we have expressed $e \notin A'$ α-recursively in D.

Claim 6. $D \leqslant_\alpha A'$.

Proof. We show that $\hat{D} \leqslant_\alpha A'$. According to the choice of $\hat{D}$ it is enough to show that $\hat{D} \leqslant_{w\alpha} A'$.

The part $e \in \hat{D}$ is a $\Sigma_2 L_\alpha$ fact and therefore trivially α-recursive in A' because of $0' \leqslant_\alpha A'$.

Concerning the part $e \notin \hat{D}$ we have according to Claim 3

$$e \notin \hat{D} \leftrightarrow \exists \gamma (\neg \exists \delta > \gamma (\langle \delta, e \rangle \notin A)) \leftrightarrow \exists \gamma (\langle p, \langle \gamma, e \rangle \rangle \notin A')$$

for some fixed parameter p. $\square$

This finishes the proof of Theorem 4.

It is tempting to define — in analogy to the definition of high and low degrees — for α-degrees $\boldsymbol{a}$:

$$\boldsymbol{a} \text{ is intermediate} : \leftrightarrow \boldsymbol{a}' = 0^{\frac{3}{2}}.$$

$0^{\frac{3}{2}}$ does not exist for every admissible α. Therefore we consider as well the following definition which makes sense for every α:

> $\boldsymbol{a}$ is intermediate: $\leftrightarrow$ $\boldsymbol{a}'$ is equal to the greatest $\Delta_2 L_\alpha$ degree and to the greatest tame $\Sigma_2 L_\alpha$ degree.

According to this second definition intermediate α-r.e. degrees exist exactly for those α where incomplete non-hyperregular α-r.e. degrees exist. Since for these α the degree $0^{\frac{3}{2}}$ is well defined we see that both definitions characterize the same class of α-r.e. degrees.

By the preceding theorems there exist hyperregular and non-hyperregular intermediate α-r.e. degrees. Further results are needed in order to see whether intermediate α-r.e. degrees are relevant for the fine structure of α-r.e. sets and degrees. In particular it would be nice to find intrinsic properties of the intermediate α-r.e. degrees (or of those α-r.e. degrees $\boldsymbol{a}$ which satisfy $\boldsymbol{a}' \leqslant 0^{\frac{3}{2}}$ respectively $\boldsymbol{a}' \geqslant 0^{\frac{3}{2}}$) which don't mention the jump. We get some first results in this direction by using recent work of A. Leggett [3].

Leggett shows that in the case $\sigma 1 p\, \alpha = \omega$ and α-r.e. degree $\boldsymbol{a}$ contains a maximal set iff $U_2^{L_\alpha} \leqslant_{w\alpha} \boldsymbol{a}'$ (where $U_2^{L_\alpha}$ is an universal $\Sigma_2 L_\alpha$ set). If $\sigma 1 p\, \alpha = \omega$ we can write $0^{\frac{3}{2}} \leqslant \boldsymbol{a}'$ instead of the latter condition according to [6].

Further Leggett shows in [3] for the larger class of α with $\sigma 2 p\, \alpha = \omega$ that an α-r.e. degree $\boldsymbol{a}$ satisfies $U_2^{L_\alpha} \leqslant_{w\alpha} \boldsymbol{a}'$ iff $\boldsymbol{a}$ contains a maximal set or $\boldsymbol{a}$ is non-hyperregular. Thus by the preceding we get for those α with $\sigma 2 p\, \alpha = \omega$ which are not Σ_2 admissible that a hyperregular α-r.e. degree $\boldsymbol{a}$ contains a maximal set iff it is intermediate. Theorem 4 shows that hyperregular intermediate α-r.e. degrees exist for all these α and so Martin's Theorem fails in these cases.

Simpson conjectured in his thesis [13] that for those α where maximal sets exists we can find in fact hyperregular ones. His conjecture is now proved for those α which satisfy $\sigma 2p\,\alpha = \omega$.

The final theorem characterizes those α which are Σ_2 admissible in terms of their degree structure. The proof is based on non-trivial results from β-recursion theory.

Theorem 5. *Assume α is admissible. Then α is Σ_2 admissible if and only if every $\Sigma_2 L_\alpha$ degree $\boldsymbol{d} \geqslant 0'$ is the jump of an incomplete α-r.e. degree.*

Proof. If $\sigma 2 \operatorname{cf} \alpha < \sigma 2p\,\alpha$, then $0''$ is not the jump of an incomplete α-r.e. degree according to Theorem 2 in [6].

If $\sigma 2p\,\alpha \leqslant \sigma 2 \operatorname{cf} \alpha < \alpha$ (i.e. α is of type (2)) we consider the weakly inadmissible structure $\mathfrak{B} := \langle L_\alpha, C\rangle$ where $C \in 0'$ is α-r.e. and regular. By Theorem 2 there exists a $\mathfrak{B}$-r.e. degree $\boldsymbol{d}$ which is not $\mathfrak{B}$-tame r.e. such that $\boldsymbol{d} <_{\mathfrak{B}} \boldsymbol{r}$ (where $\boldsymbol{r}$ is the greatest $\mathfrak{B}$-recursive degree). Therefore there exists a $\Sigma_2 L_\alpha$ degree $\boldsymbol{d}$ which is not tame $\Sigma_2 L_\alpha$ such that $0' <_\alpha \boldsymbol{d} <_\alpha 0^{\frac{3}{2}}$. If $\boldsymbol{a}$ is an incomplete α-r.e. degree then we have $0^{\frac{3}{2}} \leqslant \boldsymbol{a}'$ if $\boldsymbol{a}$ is non-hyperregular and $\boldsymbol{a}'$ is a tame $\Sigma_2 L_\alpha$ degree if $\boldsymbol{a}$ is hyperregular. Therefore we have $\boldsymbol{d} \neq \boldsymbol{a}'$ for every incomplete α-r.e. degree $\boldsymbol{a}$.

It remains to show that for Σ_2 *admissible* α every $\Sigma_2 L_\alpha$ degree $\boldsymbol{d} \geqslant 0'$ is the jump of an incomplete α-r.e. degree.

For $\alpha = \omega$ this is the jump theorem of Sacks [7].

For $\alpha > \omega$ we apply a simplified version of the construction in the proof of Theorem 4 (define $r(e, \sigma) := q(e, \sigma)$).

References

[1] S.D. Friedman and G.E. Sacks, Inadmissible recursion theory, Bull. Amer. Math. Soc. 83 (1977) 255–256.

[2] R.B. Jensen, The fine structure of the constructible hierarchy, Ann. Math. Logic 4 (1972) 229–308.

[3] A. Leggett, α-degrees of maximal α-r.e. sets, J. Symb. Logic 43 (1978) 456–474.

[4] M. Lerman and G.E. Sacks, Some minimal pairs of α-recursively enumerable degrees, Ann. Math. Logic 4 (1972) 415–422.

[5] W. Maass, Inadmissibility, tame r.e. sets and the admissible collapse, Ann. Math. Logic 13 (1978) 149–170.

[6] W. Maass, High α-recursively enumerable degrees, Proceedings of the Second Symposium on Generalized Recursion Theory Oslo 1977 (North-Holland, Amsterdam, 1978).

[7] G.E. Sacks, Recursive enumerability and the jump operator, Trans. Amer. Math. Soc. 108 (1963) 223–239.

[8] R.A. Shore, Splitting an α-recursively enumerable set, Trans. Amer. Math. Soc. 204 (1975) 65–77.

[9] R.A. Shore, The irregular and non-hyperregular α-r.e. degrees, Israel J. Math. 22 (1975) 28–41.

[10] R.A. Shore, The recursively enumerable α-degrees are dense, Ann. Math. Logic 9 (1976) 123–155.

[11] R.A. Shore, On the jump of an α-recursively enumerable set, Trans. Amer. Math. Soc. 217 (1976) 351–363.

[12] R.A. Shore, On the $\forall\exists$-sentences of α-recursion theory, Proceedings of the Second Symposium on Generalized Recursion Theory Oslo 1977 (North-Holland, Amsterdam, 1978).
[13] S.G. Simpson, Admissible ordinals and recursion theory, Thesis, MIT, Cambridge, MA, 1971.
[14] R.I. Soare, The infinite injury priority method, J. Symb. Logic 41 (1976) 513–529.