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GENERALIZED RECURSION THEORY II

HIGH α-RECURSIVELY ENUMERABLE DEGREES

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A degree $\underset{\sim}{a}$ is said to be high if $\underset{\sim}{a}' = 0''$ where $\underset{\sim}{a}'$ is the jump of $\underset{\sim}{a}$ and 0 is the degree of the empty set. Thus $0'$ is a high degree but in ordinary recursion theory (ORT) there exist as well high recursively enumerable (r.e.) degrees below $0'$ according to a theorem of Sacks [12]. The proof of this result is a very nice application of the infinite injury priority method.

It follows from the theorem of Sacks that the notion high is not trivial. Further results show that the notions high and low ($\underset{\sim}{a}$ is low if $\underset{\sim}{a}' = 0'$) are in fact important for the study of the fine structure of the r.e. degrees in ORT. The intuitive meaning is that $\underset{\sim}{a}$ is high if $\underset{\sim}{a}$ is near to $0'$ and $\underset{\sim}{a}$ is low if $\underset{\sim}{a}$ is near to 0 in the upper semilattice of the r.e. degrees. Therefore these notions are useful for the study of non-uniformity effects in this structure where one looks for theorems which hold in some regions of this semilattice but not everywhere (see e.g. Lachlan [4]).

In addition high degrees are interesting for technical reasons. Some results have been proved for high degrees and it is not yet known whether they are true for all r.e. degrees (see e.g. Cooper [1]).

Finally high degrees are a link between the structure of r.e. degrees and the structure of r.e. sets according to a theorem of Martin (see [15]): A degree contains a maximal r.e. set if and only if it is a high r.e. degree.

In α-recursion theory for admissible ordinals α the deeper properties of r.e. degrees and r.e. sets are explored in a general setting and one tries to find out which assumptions are really needed in order to do certain constructions. We refer the reader to the survey papers by Lerman and Shore in this volume for more information.

It turned out that in fact several priority arguments can be transferred to α-recursion theory (see e.g. Sacks-Simpson [14], Shore [16], Shore [18]). Other results of ORT have been proved for many admissible α but it is still open whether they hold for all admissible α (e.g. the existence of minimal pairs of α-r.e. degrees [6] ,[21] and the existence of minimal α-degrees [17],[7]). Lerman [5] closed the gap between provable existence and provable non-existence in the case of maximal α-r.e. sets.

For some time one thought that the existence of high α-r.e. degrees below $0'$ was as well completely settled by Shore [20], but an error was found in the proof of Theorem 2.3. in [20]*. The problem was then open again except for Σ_2-admissible α where the existence proof from ORT works and for α such that $0'$ is the only non-hyperregular α-r.e. degree where every α-r.e. degree below $0'$ is low according to [20] (these are the types (1) and (4) in our characterization in §3).

We close the gap in this paper by proving that high α-r.e. degrees below $0'$ exist if and only if $\sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha$. This result was not expected and is different from the result in [20]. We think that the new result is a lucky circumstance for α-recursion theory since it was thought in [20] that the situation is somewhat trivial (every non-hyperregular α-r.e. degree is high). Now it turns out that inadmissibility (in form of non-hyperregularity) influences the behaviour of the jump of an α-r.e. degree but is

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*I would like to thank R.A. Shore for informing me about this.

not so strong that it overruns everything (this will become even clearer in our forthcoming paper [11]).

The plan of this paper is as follows:

§0. contains some basic definitions and facts.

In §1. we construct high α-r.e. degrees below 0' for the case $\alpha > \sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha$. We give some motivation for the construction so that this chapter should be readable for anyone who has seen before an infinite injury priority argument in ORT (e.g.[23]). The construction reflects several typical features of α-recursion theory and uses strategies which would not work in ORT.

In §2. we prove that there exist no high α-r.e. degrees below 0' in the case $\sigma 2\mathrm{cf}\,\alpha < \sigma 2\mathrm{p}\,\alpha$ by using some basic properties of strongly inadmissible structures. Along the way some first results are proved about a distinguished degree between 0' and 0'' for which we write $0^{3/2}$.

A summary is given in §3. . Four types of admissible ordinals have to be distinguished as far as the behaviour of the jump of r.e. degrees is concerned.

§0. Preliminaries

Lowcase greek letters are always ordinals, β and λ are always limit ordinals and α is always admissible in this paper. We consider only structures $\mathfrak{B} = \langle L_\beta, B\rangle$ where $B \subseteq L_\beta$ and B is regular over L_β, i.e. $\forall \gamma < \beta (L_\gamma \cap B \in L_\beta)$. We say that a set $D \subseteq L_\beta$ is $\Sigma_n\mathfrak{B}$ if D is definable by some Σ_n formula (which may contain elements of L_β as parameters) over the structure $\mathfrak{B}$.

For $\lambda \leq \beta$ one writes $\sigma ncf^{\mathfrak{B}}\lambda$ for the least $\delta \leq \lambda$ such that some $\Sigma_n\mathfrak{B}$ function maps δ cofinally into λ and one writes $\sigma np^{\mathfrak{B}}\beta$ for the least $\delta \leq \beta$ such that some $\Sigma_n\mathfrak{B}$ function p projects β into δ (i.e. p maps β 1-1 into δ). We write $\sigma ncf\, \alpha$ instead of $\sigma ncf^{L_\alpha}\alpha$ and $\sigma np\, \alpha$ instead of $\sigma np^{L_\alpha}\alpha$.

A set $D \subseteq L_\beta$ is called $\mathfrak{B}$-r.e. ($\mathfrak{B}$-recursive) if D is $\Sigma_1\mathfrak{B}$ ($\Delta_1\mathfrak{B}$). We say that a set D is tame-$\Sigma_n\mathfrak{B}$ if the set of "positive neighborhoods" $\{K \in L_\beta \mid K \subseteq D\}$ is $\Sigma_n\mathfrak{B}$. A set $K \subseteq L_\beta$ is called β-finite if $K \in L_\beta$.

An ordinal $\delta < \beta$ is called a (regular) β-cardinal if $L_\beta \models [\delta$ is a (regular) cardinal$]$.

We fix for the following universal $\Sigma_n\mathfrak{B}$ sets $U_n^{\mathfrak{B}}$ (i.e. for every set $D \subseteq L_\beta$: D is $\Sigma_n\mathfrak{B}$ if and only if $D = \{x \mid \langle e,x\rangle \in U_n^{\mathfrak{B}}\}$ for some $e \in \beta$) which are given by some $\Sigma_n\mathfrak{B}$ definition. In the special case $n = 1$ we write $W_e^{\mathfrak{B}}$ for $\{x \mid \langle e,x\rangle \in U_1^{\mathfrak{B}}\}$.

For sets $A, D \subseteq L_\beta$ one says that A is $\mathfrak{B}$-reducible to D (written $A \leq_{\mathfrak{B}} D$) if there is some index $e \in \beta$ such that for all $K \in L_\beta$

$$K \subseteq A \leftrightarrow \exists H_1 H_2 \in L_\beta (\langle K,0,H_1,H_2\rangle \in W_e^{\mathfrak{B}} \wedge H_1 \subseteq D \wedge H_2 \subseteq L_\beta - D)$$

and

$$K \subseteq L_\beta - A \leftrightarrow \exists H_1 H_2 \in L_\beta (\langle K,1,H_1,H_2\rangle \in W_e^{\mathfrak{B}} \wedge H_1 \subseteq D \wedge H_2 \subseteq L_\beta - D).$$

The index e can be communicated by writing $A \leq_{\mathcal{L}}^{e} D$.

One further defines that A is weakly $\mathcal{L}$-reducible to D in the same way but with the sets $K \in L_\beta$ restricted to singletons $\{x\}$ (written $A \leq_{w\mathcal{L}} D$).

An equivalence relation $A =_{\mathcal{L}} D$ is defined by $A \leq_{\mathcal{L}} D \wedge D \leq_{\mathcal{L}} A$ and the equivalence classes are called <u>$\mathcal{L}$-degrees</u> . One says that a degree $\underset{\sim}{a}$ has certain properties if there exist a set $A \in \underset{\sim}{a}$ which has all these properties.

We study in this paper the <u>α-jump operator</u> (see Shore [20] for a discussion of the definition) :

$A' := \{\langle e,x\rangle \mid \exists\ H_1 H_2 \in L_\alpha (\langle x,H_1,H_2\rangle \in W_e \wedge H_1 \subseteq A \wedge H_2 \subseteq L_\alpha - A)$

is the jump of a set $A \subseteq L_\alpha$ in α-recursion theory (we always write W_e instead of $W_e^{L_\alpha}$). Since we have $A \leq_\alpha D \rightarrow A' \leq_\alpha D'$ this definition gives rise to the definition of the α-jump operator $\underset{\sim}{a} \mapsto \underset{\sim}{a}'$ for α-degrees $\underset{\sim}{a}$.

We write 0 for the α-degree of the empty set and 0'' instead of (0')' . Observe that $U_1^{L_\alpha} \in 0'$ and (using the admissibility) $U_2^{L_\alpha} \in 0''$. Furthermore we have for regular sets A that $U_1^{\langle L_\alpha, A\rangle} =_\alpha A'$.

One says that an α-r.e. set A is complete if $A \in 0'$; otherwise A is called incomplete.

We often use without further mentioning the <u>regular set theorem</u> of Sacks which says that every α-r.e. degree contains a regular α-r.e. set (see [13], [22],[8] for proofs).

For a set $A \subseteq L_\alpha$ one writes rcf A for the least $\delta \leq \alpha$ such that a cofinal function $f : \delta \rightarrow \alpha$ exists which is weakly α-reducible to A . The set A is called <u>hyperregular</u> if rcf A = α , otherwise A is called <u>non-hyperregular</u>.

Observe that we have for regular A $\mathrm{rcf}\ A = \sigma 1\mathrm{cf}^{\langle L_\alpha, A\rangle}\alpha$, in particular A is non-hyperregular iff $\langle L_\alpha, A\rangle$ is inadmissible. Hyperregularity is -contrary to regularity- a property of degrees rather than of single representativs : if $\underset{\sim}{a}$ is an α-degree then rcf A is the same for every $A \in \underset{\sim}{a}$.

Simpson proved in his thesis [22] that for any $\gamma \in \alpha$ we have that $\gamma = \mathrm{rcf}\ A$ for some α-r.e. A iff γ is a regular α-cardinal and $\sigma 2\mathrm{cf}^{L_\alpha}\gamma = \sigma 2\mathrm{cf}\ \alpha$. The following Lemma combines in b) Simpson's result with Theorem 2.1. of Shore [19]. The proofs of a) and c) are straightforward (consider a Σ_2 projection from $\varkappa$ into $\sigma 2\mathrm{p}\alpha$ for c)).

<u>Lemma 1 :</u>

a) $0'$ is a non-hyperregular α-degree iff $\sigma 2\mathrm{cf}\ \alpha < \alpha$.

b) There exists an incomplete non-hyperregular α-r.e. degree iff either $\sigma 2\mathrm{p}\alpha \le \sigma 2\mathrm{cf}\ \alpha < \alpha$ or $\sigma 2\mathrm{cf}\ \alpha < \sigma 2\mathrm{p}\alpha < \alpha$ and there is a regular α-cardinal $\varkappa \ge \sigma 2\mathrm{p}\alpha$ such that $\sigma 2\mathrm{cf}^{L_\alpha}\varkappa = \sigma 2\mathrm{cf}\ \alpha$.

c) We have $\sigma 2\mathrm{cf}^{L_\alpha}\varkappa = \sigma 2\mathrm{cf}\ \alpha$ for every regular α-cardinal $\varkappa$ such that $\sigma 2\mathrm{p}\alpha < \varkappa$ and $\sigma 2\mathrm{cf}\ \alpha < \varkappa$.

Finally define for any structure $\mathcal{L} = \langle L_\beta, B\rangle$

$\rho_{n,\beta}^{\mathcal{L}} := \mu\delta \le \beta$(a $\Sigma_n\ \mathcal{L}$ set $M \subseteq \delta$ exists such that $M \notin L_\beta$)

(we write $\rho_{n,\beta}$ instead of $\rho_{n,\beta}^{L_\beta}$).

According to Jensen's Uniformization Theorem [2] we have $\rho_{n,\beta} = \sigma n\mathrm{p}\beta$ for every $n > 0$ and every limit ordinal β.

We will often use without further mentioning the equalities $\rho_{n,\alpha} = \sigma n\mathrm{p}\alpha$ for $n = 1,2$ which are easier to show because Σ_2-uniformization is trivial for admissible α.

We refer the reader to Devlin [2] for all details concerning constructibility.

§1. Construction of high α-r.e. degrees

At first we sketch the construction of incomplete high r.e. sets in ORT. The original proof is due to Sacks [12]. Additional ideas of Lachlan and Soare are used in the very perspicuous version of the construction as it is presented in Soare[23] (we refer the reader to this paper for more motivation and details concerning the proof in ORT).

In order to bring the requirement $A' \in 0''$ in the reach of a recursive construction we associate with a fixed Σ_2 set $S \in 0''$ a r.e. set B_S which is defined by

$\langle e,y\rangle \in B_S \leftrightarrow \forall y' \leq y \exists z \neg\phi(e,y',z)$

where $\exists y \forall z \phi(\cdot,y,z)$ is a fixed Σ_2 definition of S over L_ω. Then we have for every $e \in \omega$ $1-S(e) = \lim_{y\to\omega} B_S(\langle e,y\rangle)$ and it is enough to insure that for all $e \in \omega$ $\lim_{y\to\omega} A(\langle e,y\rangle) \simeq \lim_{y\to\omega} B(\langle e,y\rangle)$ in order to get $A' \in 0''$. So for every e we set up a positive requirement $P_e : \lim_{y\to\omega} A(\langle e,y\rangle) = \lim_{y\to\omega} B(\langle e,y\rangle)$.
P_e is a requirement which is hard to satisfy if $e \notin S$ since in this case we have to put all but finitely many elements of $\{\langle e,y\rangle \mid y \in \omega\}$ into A.

A conflict arises because we have to satisfy as well for all $e \in \omega$ $N_e : \neg C = \Phi_e(A)$ where $C \in 0'$ is a fixed r.e. set. The requirements N_e are satisfied by preserving a disagreement between $C_t(x)$ and $\Phi_{e,t}(A_t,x)$ for a suitable argument x and by forcing the appearance of such a disagreement (respectively of an argument x such that $\Phi_e(A,x)\uparrow$) on the way of preserving as well agreements between $C_t(x)$ and $\Phi_{e,t}(A_t,x)$ for all x out of an initial segment γ of ω which is chosen as long as possible $(\gamma \leq \omega)$.(We have written $\Phi_e(A,\cdot)$ for the function which is partially recursive in A with index e.)

This strategy of preservation in order to get $\neg C \leq^e A$ is on first sight contrary to intuition. But if we preserve -as soon as it appears during the construction- agreement between $C_t(x)$ and $\phi_{e,t}(A_t,x)$ for every $x \in \gamma$ we actually try to make $C\restriction\gamma$ recursive. Since $C = C\restriction\omega$ is not recursive we must then have $\gamma < \omega$ and therefore $\neg\phi_e(A,\gamma) \simeq C(\gamma)$.

We write I_e for the injury set of N_e which is the set of all elements x that are put into A -as demanded by some positive requirement $P_{e'}$ with $e' < e$ - although they destroy a computation in A which should be preserved in order to satisfy N_e . Then we can be a little more exact in our description of the preservation strategy and say that although we try to make $C\restriction\gamma$ recursive we only succeed in making $C\restriction\gamma$ recursive in I_e (γ as before). But we can still get the wanted conclusion $\gamma < \omega$ even if I_e is infinite since we need only $C \nleq I_e$ for the argument above. Since I_e is recursive in $\{\langle e',y\rangle \in A \mid e' < e \wedge y \in \omega\}$ we can prove $C \nleq I_e$ for every $e \in \omega$ during the inductive argument where one shows that for every $e \in \omega$ $\neg C \leq^e A$ and $\lim_{y\to\omega} A(\langle e,y\rangle) = \lim_{y\to\omega} B(\langle e,y\rangle)$ (we use here that $C \nleq \{\langle e',y\rangle \in B \mid e' < e \wedge y \in \omega\}$ for every e).

Observe that in writing $C(\gamma)$ etc. we have followed the usual convention to identify a set with its characteristic function.

The construction from ORT works as well for Σ_2 admissible α (Shore [20]). But there are several reasons why this construction does not work for the other admissible α . We discuss five of these problems in the following and we simultaneously try to motivate the new features of the subsequent construction for the case $\alpha > \sigma 2cf\alpha \geq \sigma 2p\alpha$.

1) Assume that we succeed in constructing the set A in such

a way that $\forall e \in \alpha\,(A^{(e)} =^* B_S^{(e)})$ with S and B_S as before (define for any set M : $M^{(e)} := M \cap (\{e\} \times L_\alpha)$; $M_1 =^* M_2$ means that $M_1 - M_2 \in L_\alpha$ and $M_2 - M_1 \in L_\alpha$).

This doesn't imply in general that $S \leq_\alpha A'$ if $\sigma 2\mathrm{cf}\,\alpha < \alpha$. We have of course for every $e \in \alpha$ that $e \in S \leftrightarrow$
$\exists y_e \forall y \geq y_e (\neg \langle e,y \rangle \in A) \leftrightarrow \exists y_e (\neg \langle p,e,y_e \rangle \in A')$
for some fixed parameter p. But if we want to reduce in the same way questions "$K \subseteq S$" to questions about A' we need the existence of a bound for the set $\{y_e \mid e \in K\}$ of witnesses. Since $S \in 0''$ can't be tame-Σ_2 L_α if α is not Σ_2 admissible (see §2.) we can hardly expect that this bound exists for all α-finite K such that $K \subseteq S$.

We overcome this difficulty by using in a positive way that α is not Σ_2 admissible. For these α there exist non-hyperregular α-r.e. sets and in the case $\alpha > \sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha$ there exist even incomplete non-hyperregular α-r.e. sets according to Shore [19]. But for non-hyperregular A we can avoid the search for witnesses y_e : Take a cofinal function $f : \mathrm{rcf}\,A \to \alpha$ which is weakly α-recursive in A. Then we have
$e \in S \leftrightarrow \forall x \in \mathrm{rcf}\,A\ \exists y\, z (y = f(x) \wedge z \geq y \wedge \neg\langle e,z \rangle \in A) \leftrightarrow$
$\{p\} \times \mathrm{rcf}\,A \times \{e\} \subseteq A'$
for some fixed parameter p which implies that for every α-finite K we have
$K \subseteq S \leftrightarrow \{p\} \times \mathrm{rcf}\,A \times K \subseteq A'$.

Convention: We say "α-recursive in" and "weakly α-recursive in" for "$\leq_\alpha$" respectively "$\leq_{w\alpha}$" as usual. But there is a problem with this interpretation, see [9].

2) For the considered α where $\alpha > \sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha$ it can happen that $0''$ does not contain a regular Σ_2 L_α set. According to [11] this occurs if and only if $\sigma 3\mathrm{cf}\,\alpha < \sigma 3\mathrm{p}\,\alpha$. We will con-

struct in [11] an α where $\sigma 3\mathrm{cf}\,\alpha < \sigma 3\mathrm{p}\,\alpha \leq \sigma 2\mathrm{p}\,\alpha < \sigma 2\mathrm{cf}\,\alpha < \alpha$. This example is the most difficult one with respect to our construction of incomplete high α-r.e. degrees since $0''$ does not contain a regular Σ_2 set and we have $\sigma 2\mathrm{p}\,\alpha < t\sigma 2\mathrm{p}\,\alpha$ (see [6] for the definition of the tame Σ_2 projectum $t\sigma 2\mathrm{p}\,\alpha$).

3) In consequence of the preceding the plan for our construction is as follows : We take a fixed incomplete non-hyperregular α-r.e. set D and make sure that $A^{(0)} =^* D$ in order to make A non-hyperregular. Further for $e > 0$ we want to have that $A^{(e)} =^* B_S^{(e)}$. As before we set up for every $e \in \alpha$ a positive requirement P_e which tries to satisfy this condition concerning $A^{(e)}$.

It is crucial for the infinite injury argument that the set of those elements which should be put into A in order to satisfy all requirements out of an initial segment of the priority list is not too complicated. According to point 2) this forces us to make our priority list no longer than $\sigma 2\mathrm{p}\,\alpha$ because only for α-finite sets K of α-cardinality less than $\sigma 2\mathrm{p}\,\alpha$ it is guaranteed that $B_S \cap K \times L_\alpha$ is α-recursive. It is not easy to work with such a short priority list in an infinite injury construction since the α-recursive approximation to this list is very weak if $\sigma 2\mathrm{p}\,\alpha < \sigma 2\mathrm{cf}\,\alpha$. We introduce a clause b) in the construction which makes it possible to control in many situations those unwanted injuries which are merely due to bad guessing of priorities.

4) We want to prove by induction on the priority $p(e)$ that for every e we have $A^{(e)} =^* B^{(e)}$. There is a problem in the case that $p(e)$ is a limit ordinal since the induction hypothesis doesn't imply then that $\bigcup \{ A^{(i)} \mid p(i) < p(e)\} =^* \bigcup \{B^{(i)} \mid p(i) < p(e)\}$ and we can't control the degree of the injury set

I_e . We use the fact that this situation is only possible if $\sigma 2\mathrm{cf}\,\alpha > \omega$ since $\sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha$. $\sigma 2\mathrm{cf}\,\alpha > \omega$ implies that there are enough fixpoint stages in the construction so that it is in fact not necessary to determine the degree of the injury set I_e .

5) There is a problem with the preservation strategy of Sacks in the case that there are non-hyperregular injury sets I_e (which will occur in our construction since $A^{(0)}$ is non-hyperregular). If we want to preserve then agreements $C(x) = \phi_e(A,x)$ for $x < \gamma$ these computations may altogether use an unbounded part of A even if $\gamma < \alpha$. Since this would endanger the positive requirements of lower priority we have to be much more careful with preservations. For this sake we introduce "e-fixpoints" in the case $\sigma 2\mathrm{cf}\,\alpha > \omega$. In the case $\sigma 2\mathrm{cf}\,\alpha = \omega$ we divide α into rcf D many blocks as in Shore [18] (doing the same thing in the case $\sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha > \omega$ would be troublesome because of limit points in the priority list).

<u>Theorem 1 :</u> Assume that $\alpha > \sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha$. If C and D are α-r.e. sets such that $C \not\leq_\alpha D$ and D is non-hyperregular then there exists an α-r.e. set A such that $D \leq_\alpha A$, $C \not\leq_\alpha A$ and $A' =_\alpha 0''$.

The rest of this chapter is devoted to the <u>proof</u> of this theorem. After some preparations we will describe the construction of the set A for the case $\sigma 2\mathrm{cf}\,\alpha > \omega$. We will show in the Lemmata 3,4,5 that this set A has the properties we want. The construction for the case $\sigma 2\mathrm{cf}\,\alpha = \omega$ is rather close to the construction in ORT and will be discussed briefly afterwards.

We fix for the following regular α-r.e. sets $C,D \subseteq \alpha$ such that $C \not\leq_\alpha D$ and D is non-hyperregular. $(C_\sigma)_{\sigma<\alpha}$ and $(D_\sigma)_{\sigma<\alpha}$ are in the following fixed α-recursive enumerations of these sets.

Take a Σ_2 L_α set $S \subseteq \alpha$ such that $S \in 0''$ and fix a Δ_0 formula Ψ such that $\beta \in S \leftrightarrow L_\alpha \models \exists \gamma \, \forall x \, \Psi(\beta,\gamma,x)$.
Define the α-r.e. set $B \subseteq \alpha \times \alpha$ as follows :

$$\langle\beta,\gamma\rangle \in B :\leftrightarrow ((\beta = 0 \wedge \gamma \in D) \vee (\beta > 0 \wedge L_\alpha \models \forall \gamma' \leq \gamma \, \exists x \, \neg\Psi(\beta,\gamma',x))) .$$

Then we have for $\beta > 0$:

$$\beta \in S \rightarrow \{\gamma \mid \langle\beta,\gamma\rangle \in B\} = \mu\delta(L_\alpha \models \forall x \, \Psi(\beta,\delta,x)) < \alpha \quad \text{and}$$

$$\neg\beta \in S \rightarrow \{\gamma \mid \langle\beta,\gamma\rangle \in B\} = \alpha \quad .$$

Fix an α-recursive enumeration $(B_\sigma)_{\sigma<\alpha}$ of B for the following.

For any set M and any $x \in L_\alpha$ we will use in the following $M^{(x)}$ as an abbreviation for $M \cap (\{x\} \times L_\alpha)$.

A_σ will be the set of elements which have been put into A <u>before</u> stage σ .

<u>Lemma 2 :</u> Assume that K is an α-finite set of α-cardinality less than $\sigma 2 \mathrm{cf}\alpha$. Further assume that W is an α-r.e. set such that $W^{(x)}$ is regular for every $x \in K$. Then $\bigcup \{ W^{(x)} \mid x \in K \}$ is regular.

<u>Proof :</u> Fix an enumeration $(W_\sigma)_{\sigma<\alpha}$ of W . For given $\beta < \alpha$ define a Σ_2 function $f : K \rightarrow \alpha$ by $f(x) := \mu\sigma(W_\sigma \cap W^{(x)} \cap L_\beta = W^{(x)} \cap L_\beta)$. There exists a bound σ_0 for $\mathrm{Rg}\, f$ and we have $\bigcup \{ W^{(x)} \mid x \in K\} \cap L_\beta \subseteq W_{\sigma_0}$.

In the following we will write $x \in W_{e,\sigma}$ for $L_\sigma \models \phi(\langle e,x\rangle)$ where ϕ is a fixed Σ_1 L_α definition of $U_1^{L_\alpha}$.

For the considered α-r.e. sets A and C and their enumerations $(A_\tau)_{\tau<\alpha}$ and $(C_\tau)_{\tau<\alpha}$ we will say that <u>at stage σ there exists a computation of "$C \leq^e A$" for "$K \subseteq L_\alpha - C$" with negative neighborhood H</u> if $\exists H' \in L_\sigma(\langle K,1,H',H\rangle \in W_{e,\sigma} \wedge H' \subseteq A_\sigma \wedge H \subseteq L_\sigma - A_\sigma)$.

<u>Case i) : $\alpha > \sigma 2cf\alpha \geq \sigma 2p\alpha$ and $\sigma 2cf\alpha > \omega$.</u>

The next definition is the fixpoint device which was mentioned in point 5) of the motivation.

λ is an <u>e-fixpoint</u> at stage $\beta \geq \lambda$:$\leftrightarrow$
for every $\tau < \lambda$ there is a τ' such that $\tau \leq \tau' < \lambda$ and there is a stage $\sigma < \lambda$ such that at stage σ there exists a computation of "$C \leq^e A$" for "$\tau' - C_\lambda \subseteq L_\alpha - C$" with negative neighborhood H and we have $H \subseteq L_\alpha - A_\beta$.

We say that this e-fixpoint λ is <u>inactive</u> at stage β if $C_\lambda \cap \lambda \neq C_\beta \cap \lambda$.

The <u>"restraint function" $r : \alpha \times \alpha \to \alpha$</u> will play a similar role as in Soare [23] and is defined by cases :

<u>Case 1):</u> There exists a stage $\sigma \leq \beta$ such that some $\lambda < \sigma$ is an inactive e-fixpoint at all stages in $[\sigma,\beta] := \{\tau \mid \sigma \leq \tau \leq \beta\}$. Take the least such σ . Define $r(e,\beta)$ to be the least $\lambda < \sigma$ which is an inactive e-fixpoint at all stages in $[\sigma,\beta]$.

<u>Case 2):</u> Define $r(e,\beta)$ to be the union of all e-fixpoints λ at β otherwise.

We fix an 1-1 Σ_2 L_α function g which maps $\sigma 2p\alpha$ partially onto α . $g^{-1}(e)$ will be the priority of the requirements P_e, N_e for $e \in \alpha$. Using the assumption $\sigma 2p\alpha \leq \sigma 2cf\alpha$ it is easy to see that $g \upharpoonright (\gamma \cap \text{dom } g)$ is α-finite and $B^{(<\gamma)} \leq_\alpha D$ for every $\gamma < \sigma 2p\alpha$ where $B^{(<\gamma)} := \bigcup \{B^{(e)} \mid g^{-1}(e) < \gamma\}$.

We further need an α-recursive approximation function $g^{\cdot}(\cdot)$ (of two arguments) with α-recursive domain which has the property that for all $\gamma < \sigma 2p\alpha$ there exists an ordinal $\tau_\gamma < \alpha$ such that for all $x \in \gamma \cap \mathrm{dom}\, g$ and all $\tau \geq \tau_\gamma$ we have $g^\tau(x) \simeq g(x)$.
In addition we want to have that

(1) $g(x)\downarrow \leftrightarrow \exists \sigma_0 \forall \sigma \geq \sigma_0 (g^\sigma(x)\downarrow)$ and

(2) $\forall$ limits $\lambda < \alpha (g^\lambda(x)\downarrow \leftrightarrow \exists \sigma_0 < \lambda \forall \sigma (\sigma_0 \leq \sigma < \lambda \rightarrow g^\sigma(x)\downarrow))$

and that $g^\sigma(\cdot)$ is 1-1 for every $\sigma < \alpha$.
Because of the distinguished role of the requirement P_0 we further need that $g(0) \simeq 0$ and $g^\sigma(0) \simeq 0$ for all $\sigma < \alpha$.

The definition of an approximation function $g^{\cdot}(\cdot)$ with these properties is routine.

Observe that in general we can't get the following property which one would really like to have :
$\forall \gamma < \sigma 2p\alpha \exists \sigma_0 \forall z \leq \gamma \forall \sigma \geq \sigma_0 (g^\sigma(z) \simeq g(z))$
(see the points 2) and 3)).

<u>Construction :</u>

At stage σ we consider every $\langle\beta,\gamma\rangle \in B_{\sigma+1}$ such that $g^\sigma(z) \simeq \beta$ for some $z < \sigma 2p\alpha$.
If $\langle\beta,\gamma\rangle$ is not already an element of A_σ we put $\langle\beta,\gamma\rangle$ into A at stage σ if

a) $\langle\beta,\gamma\rangle \geq r(g^\sigma(z'),\sigma)$ for all $z' \in (z+1) \cap \mathrm{dom}\, g^\sigma$ and

b) $\langle\beta,\gamma\rangle \geq \tau$ for all $\tau < \sigma$ such that not $(z+1) \cap \mathrm{dom}\, g^\sigma \subseteq (z+1) \cap \mathrm{dom}\, g^\tau$.

End of construction.

For a negative requirement N_e there are in general unboundedly many stages σ where some positive requirement $P_{e'}$ of truly lower priority (i.e. $g^{-1}(e) \leq g^{-1}(e')$) thinks that it may injure N_e because of the weak approximation property of $g^{\cdot}(\cdot)$. The following Lemma shows that in some spécial situations these unwanted injuries will not occur because of clause b) in the construction.

In the following we always write τ_γ for the least τ such that $\forall \sigma \geq \tau \, \forall x \in \gamma \cap \text{dom } g \, (g^\sigma(x) \simeq g(x))$.

Lemma 3 : Assume that $\gamma < \sigma 2p\alpha$, $\tau_\gamma \leq \sigma \leq \tau$, $\gamma \cap \text{dom } g^\sigma = \gamma \cap \text{dom } g$ and $z \in \gamma \cap \text{dom } g$. If at stage τ an element $\langle\beta,\delta\rangle$ is put into A such that $\langle\beta,\delta\rangle < \sigma$ and $\langle\beta,\delta\rangle < r(g^\tau(z),\tau)$ then there exists a $z' < z$ such that $z' \in \gamma \cap \text{dom } g$ and $g(z') \simeq \beta$.

Proof : According to the construction there exists a z' such that $g^\tau(z') \simeq \beta$. Since clause b) does not restrain $\langle\beta,\delta\rangle$ at stage τ we have $(z'+1) \cap \text{dom } g^\tau \subseteq (z'+1) \cap \text{dom } g^\sigma$. We further have $z' < z$ because $\langle\beta,\delta\rangle < r(g^\tau(z),\tau)$ and $\langle\beta,\delta\rangle$ is not restrained by clause a) at stage τ . Since $\tau \geq \sigma \geq \tau_\gamma$ it follows that $(z'+1) \cap \text{dom } g^\tau = (z'+1) \cap \text{dom } g$ and $g^\tau \restriction ((z'+1) \cap \text{dom } g) = g \restriction ((z'+1) \cap \text{dom } g)$. In particular we have that $z' \in \text{dom } g$ and $g(z') \simeq g^\tau(z') \simeq \beta$.

The following Lemma will solve the problem which was described in point 3) of the motivation : In the case where the priority $g^{-1}(e)$ of some negative requirement N_e is a limit ordinal we have problems to control $\bigcup \{A^{(i)} \mid g^{-1}(i) < g^{-1}(e)\}$ and the injury set I_e . Lemma 4 gives a sufficient condition for a stage σ that some computation which exists at stage σ will not be destroyed later. It is important that this condition can be expressed by using just $\bigcup\{B^{(i)} \mid g^{-1}(i) < g^{-1}(e)\}$, not $\bigcup\{A^{(i)} \mid g^{-1}(i) < g^{-1}(e)\}$. A fixpoint argument in Lemma 5 will show that this condition will be met by an unbounded set of stages.

The properties of $\sigma 2p\alpha$ and the assumption $\sigma 2p\alpha \leq \sigma 2cf\alpha$ imply that for every $\gamma < \sigma 2p\alpha$ there exists a stage $\tau \geq \tau_\gamma$ such that for every $z \in \gamma \cap \text{dom } g$

$\exists \delta \exists \lambda($ λ is an inactive $g(z)$-fixpoint at all stages $\sigma \geq \delta$ $) \rightarrow$

$\exists \lambda($ λ is an inactive $g(z)$-fixpoint at all stages $\sigma \geq \tau$ $)$.

In the following we write τ_γ' for the least such $\tau \geq \tau_\gamma$.

We further define $B^{(<\gamma)} := \bigcup\{B^{(e)} \mid g^{-1}(e) < \gamma\}$ and $B_\sigma^{(<\gamma)} := B^{(<\gamma)} \cap B_\sigma$.

<u>Lemma 4 :</u> Assume that $\sigma \geq \tau_\gamma'$ is a stage such that $\gamma \cap \text{dom } g^\sigma = \gamma \cap \text{dom } g$, $B_\sigma^{(<\gamma)} \cap \sigma = B^{(<\gamma)} \cap \sigma$ and no element $x < \sup\{r(g(z),\sigma) \mid z \in \gamma \cap \text{dom } g\}$ is put into A at stage σ . Then we have for every $z \in \gamma \cap \text{dom } g$ and for every stage $\tau \geq \sigma$: $r(g(z),\tau) \geq r(g(z),\sigma)$ and no element $x < \sigma$ with $x < r(g(z),\tau)$ is put into A at stage τ .

<u>Proof :</u> Induction on $z \in \gamma \cap \text{dom } g$.

Assume for a contradiction that some $x < \sigma$ with $x < r(g(z), \sigma_0)$ is put into A at stage σ_0 .

By Lemma 3 there is some $z' < z$ such that $z' \in \text{dom } g$ and $g(z') \simeq \beta$ where $x = \langle\beta,\delta\rangle$ for some δ . Therefore we have $x \in B^{(<\gamma)} \cap \sigma = B_\sigma^{(<\gamma)} \cap \sigma$. We consider two cases :

<u>a)</u> x was not put into A at stage σ since $x < r(g(z''),\sigma)$ where $z'' \in (z'+1) \cap \text{dom } g$. By our induction hypothesis we have $r(g(z''),\sigma) \leq r(g(z''),\sigma_0) = r(g^{\sigma_0}(z''), \sigma_0)$ and x will not be put into A at stage σ_0 either.

<u>b)</u> x was not put into A at stage σ since there exists $\sigma' < \sigma$ such that $\sigma' > x$ and not $(z'+1) \cap \text{dom } g^\sigma \subseteq (z'+1) \cap \text{dom } g^{\sigma'}$.

Since $(z'+1) \cap \mathrm{dom}\, g^{\sigma} \subseteq (z'+1) \cap \mathrm{dom}\, g^{\sigma_0}$ x is not put into A at stage σ_0 because of clause b) in the construction.

It remains to prove that $r(g(z),\tau) \geq r(g(z),\sigma)$ for all $\tau \geq \sigma$. Assume that there is a minimal stage $\sigma_0 > \sigma$ such that $r(g(z),\sigma_0) < r(g(z),\sigma)$. By the preceding no element $y < r(g(z),\sigma)$ will be put into A at some stage τ where $\sigma \leq \tau \leq \sigma_0$. $r(g(z),\sigma_0) < r(g(z),\sigma)$ is therefore only possible if $r(g(z),\sigma_0)$ is defined according to case 1) of the definition of r whereas $r(g(z),\sigma)$ is defined according to case 2). Since $r(g(z),\sigma_0) < \sigma$ no element $y < r(g(z),\sigma_0)$ will be put into A at any stage $\tau \geq \sigma_0$: Otherwise assume that σ_1 is the minimal such τ. Since $r(g(z),\sigma_0)$ is defined according to case 1) we have $r(g^{\sigma_1}(z),\sigma_1)$ $= r(g(z),\sigma_0) > y$ and y can't be put into A at stage σ_1 as it was shown in the first part of this proof. Thus we have proved that some $\lambda < \sigma_0$ is an inactive g(z)-fixpoint at all stages in $[\sigma_0,\alpha)$ whereas there is no inactive g(z)-fixpoint at stage σ. Since we have $\tau_{\gamma}' \leq \sigma$ this gives a contradiction to the definition of τ_{γ}' and we have proved that $r(g(z),\tau) \geq r(g(z),\sigma)$ for all $\tau \geq \sigma$.

<u>Remark:</u> If σ satisfies the assumptions of Lemma 4 then no element $x < \sup\{r(g(z),\sigma) \mid z \in \gamma \cap \mathrm{dom}\, g\}$ is put into A at any stage $\tau \geq \sigma$. Therefore these stages σ play a role in this proof which is similar to the role of "true stages" (see Soare[23]) in the proof in ORT.

<u>Lemma 5 :</u> For every $e \in \alpha$ we have

a) $\neg C \leq_{\alpha}^{e} A$ and

b) $A^{(e)} =^* B^{(e)}$.

<u>Proof :</u> For convenience we prove a) and b) simultaneously by

induction on $g^{-1}(e)$. Assume for the following that $g^{-1}(e) = z$ and that a) and b) are true for all e' such that $g^{-1}(e') < z$. Observe that this assumption does in general not imply that $\bigcup\{A^{(e')} \mid g^{-1}(e') < z\} =^* \bigcup\{B^{(e')} \mid g^{-1}(e') < z\}$ if we have $\sigma 3 \mathrm{cf}\, \alpha \le z$, which is of course possible (see point 3) of the motivation). But we get the information that $\bigcup\{A^{(e')} \mid g^{-1}(e') < z\}$ is regular : Since every $B^{(e')}$ is regular we get from $A^{(e')} =^* B^{(e')}$ that every $A^{(e')}$ is regular as well. Then Lemma 2 implies that $\bigcup\{A^{(e')} \mid g^{-1}(e') < z\}$ is regular. This is the only fact which we use from our induction hypothesis so that in the case $\sigma 1 p \alpha = \alpha$ we don't need an induction at all (this is rather surprising if compared with the situation in ORT , see [23]).

For $\gamma := z+1$ we write M for the set of those stages σ where the assumptions of Lemma 4 are satisfied. We want to prove that M is unbounded in α by using the regularity of $A^{(<z)} := \bigcup\{A^{(e')} \mid g^{-1}(e') < z\}$.

For $\lambda_n < \alpha$ define $\lambda_{n+1} := \mu\tau > \lambda_n\, (\forall y \in (((z+1) \cap \mathrm{dom}\, g^{\lambda_n}) - \mathrm{dom}\, g)\, \exists \tau' \le \tau\, (g^{\tau'}(y)\uparrow) \wedge B_\tau^{(<\gamma)} \cap \lambda_n = B^{(<\gamma)} \cap \lambda_n \wedge A_\tau^{(<z)} \cap \lambda_n = A^{(<z)} \cap \lambda_n)$.

By using property (1) of $g^{\cdot}(\cdot)$ and the fact that $B^{(<\gamma)}$ and $A^{(<z)}$ are regular α-r.e. sets it is easy to see that $\lambda_{n+1} < \alpha$ exists. For every given $\lambda_0 < \alpha$ with $\lambda_0 > \tau_\gamma'$ define then $\lambda := \sup\{\lambda_n \mid n \in \omega\}$. We have $\lambda < \alpha$ since the function $n \mapsto \lambda_n$ is $\Sigma_2\, L_\alpha$. It follows from property (2) of the appoximating function and Lemma 3 that $\lambda \in M$.

We write IN for the α-finite set of all $z' \le z$ such that some $\lambda < \alpha$ is an inactive $g(z')$-fixpoint in $[\tau,\alpha)$ for some $\tau < \alpha$ ($[\tau,\alpha) := \{\tau' \mid \tau \le \tau' < \alpha\}$) . Then we have that $r(g(z'),\cdot)$ is constant in $[\sigma_0,\alpha)$ for every $z' \in$ IN according to Lemma 4 , where σ_0 is the least element of M. Therefore it is enough to show $\sup\{r(g(z'),\sigma) \mid \sigma \in M \wedge z' \in ((z+1) \cap \operatorname{dom} g - \mathbb{N})\} < \alpha$ in order to prove that $\sup\{r(g(z'),\sigma) \mid \sigma \in M \wedge z' \in (z+1) \cap \operatorname{dom} g\} < \alpha$.

Thus assume for a contradiction that

$\forall \tau < \alpha \exists \sigma \in M \exists z' \in ((z+1) \cap \operatorname{dom} g - \mathbb{N})(\tau < r(g(z'),\sigma))$.

This implies that for every $K \in L_\alpha$

$$K \subseteq L_\alpha - C \leftrightarrow \exists \sigma \in M \exists z' \in ((z+1) \cap \operatorname{dom} g - \mathbb{N})$$
$$(\sup K < r(g(z'),\sigma) \wedge K \subseteq L_\alpha - C_\sigma) .$$

The part "$\rightarrow$" of this equivalence is obvious from our assumptions. For a proof of "$\leftarrow$" we assume that σ, z', K do satisfy the right side. By Lemma 4 we have that $r(g(z'),\sigma)$ is defined according to case 2) of the definition of r since $\neg z' \in \mathbb{N}$. Therefore there is at σ some $g(z')$-fixpoint $\lambda \leq r(g(z'),\sigma)$ such that $\sup K < \lambda$ and λ is not an inactive $g(z')$-fixpoint at σ which means that $C_\sigma \cap \lambda = C_\lambda \cap \lambda$. By Lemma 4 there is no stage $\tau \geq \sigma$ such that an element $y < \lambda$ is put into A at stage τ. Therefore there is no $y < \lambda$ and $\tau \geq \sigma$ such that $y \in C_{\tau+1} - C_\tau$ since otherwise some $\lambda' \leq \lambda$ would be an inactive $g(z')$-fixpoint in $[\tau,\alpha)$, contradicting $\neg z' \in \mathbb{N}$. Thus we have proved that $C_\sigma \cap \lambda = C \cap \lambda$ which shows that $K \subseteq L_\alpha - C$.

The equivalence which was just proved implies that $C \leq_\alpha B^{(<\gamma)}$ since "$\sigma \in M$" can be expressed α-recursively in $B^{(<\gamma)}$. But this is absurd because we have $B^{(<\gamma)} \leq_\alpha D$. Thus we have proved that $S := \sup\{r(g(z'),\sigma) \mid \sigma \in M \wedge z' \in (z+1) \cap \operatorname{dom} g\} < \alpha$.

In order to prove a) assume for a contradiction that $C \leq_\alpha^e A$. For $\lambda_n' < \alpha$ we define $\lambda_{n+1}' < \alpha$ by

$\lambda_{n+1}' := \mu\tau > \lambda_n'$((the same as in the definition of λ_{n+1}) $\wedge$ $C_\tau \cap \lambda_n' = C \cap \lambda_n' \wedge$ (at stage τ there exists a computation of

"$C \leq_\alpha^e A$" for "$\lambda_n' - C \subseteq L_\alpha - C$" with negative neighborhood H such that $H \subseteq L_\alpha - A$)) .

We have again that $\lambda' := \sup\{\lambda_n' \mid n \in \omega\} < \alpha$ for every given $\lambda_0' < \alpha$ since the function $n \mapsto \lambda_n'$ is Σ_2 definable and if we start with some $\lambda_0' > \tau_\gamma'$ it is obvious that $\lambda' \in M$ and λ' is a g(z)-fixpoint at λ' . Now it can't be the case that $r(g(z),\sigma)$ is defined for some $\sigma \in M$ according to case 1) of the definition of r because this contradicts $C \leq_\alpha^e A$ (use Lemma 4) . This implies that $r(g(z),\lambda') = \lambda'$ for all these stages $\lambda' \in M$ which is absurd since we have proved just before that $S < \alpha$.

For the proof of b) we choose $\sigma_1 \in M$ such that $\sigma_1 > S$. It follows from Lemma 4 that $A^{(e)} \cap \sigma_1 = A^{(e)}_{\sigma_1+1} \cap \sigma_1$. Further we have $A^{(e)} - \sigma_1 = B^{(e)} - \sigma_1$ by the definition of S which together shows that $A^{(e)} =^* B^{(e)}$.

The proof of Theorem 1 is now very easy. We get $\neg C \leq_\alpha A$ and $D = B^{(0)} =_\alpha A^{(0)} \leq_\alpha A$ from Lemma 5 . In order to show $S \leq_\alpha A'$ we fix a cofinal function $f : \mathrm{rcf}\, A \to \alpha$ which is weakly α-recursive in A (A is non-hyperregular since D is non-hyperregular and $D \leq_\alpha A$). Lemma 5 b) implies that for every $\beta \in \alpha$

$\beta \in S \leftrightarrow \forall x < \mathrm{rcf}\, A\ \exists \delta > f(x)\ (\neg \langle\beta,\delta\rangle \in A)$

(we may assume without loss of generality that $0 \in S$) .

There is a parameter $p \in \alpha$ such that for all β and x

$x < \mathrm{rcf}\, A \wedge \exists \delta > f(x)\ (\neg\langle\beta,\delta\rangle \in A) \leftrightarrow \langle p,x,\beta\rangle \in A'$. Then we have $K \subseteq S \leftrightarrow \{p\} \times \mathrm{rcf}\, A \times K \subseteq A'$.

Concerning the computation for "$K \subseteq L_\alpha - S$" we observe that "$K \subseteq L_\alpha - S$" can be written as a Π_2 formula. Since we have $U_2^{L_\alpha} \leq_{w\alpha} A'$ (see the first part of the proof of Theorem 2 b) in §2.) this Π_2 fact can be expressed α-recursively in A' .

Case ii) : $\alpha > \sigma 2cf\alpha = \sigma 2p\alpha = \omega$.

The proof of Theorem 1 is simpler in this case since the problem at limits of the priority list doesn't occur (see point 4) of the motivation). The construction is closer to the one in ORT [23] but we have to be aware of the other points in the motivation and the fact that we can't use the regularity of A as it is done in ORT ("true stages").

According to point 5) of the motivation we fix a strictly increasing cofinal function $f : rcf\ D \to \alpha$ which is weakly α-recursive in $B^{(0)}$ with an index e. We define then

$f^{\sigma}(x)\downarrow \;:\leftrightarrow\; \exists \tau \leq \sigma\, \exists y\, \exists H\, (\langle x,y,H\rangle \in W_{e,\tau} \wedge H \subseteq \{0\} \times L_{\alpha} - A^{(0)}_{\sigma})$.

If $f^{\sigma}(x)\downarrow$ we go to the least such $\tau \leq \sigma$ and choose $\langle x,\hat{y},\hat{H}\rangle$ minimal (with respect to a fixed canonical Δ_1 L_α well ordering $\prec_\alpha$ of L_α) such that $\langle x,\hat{y},\hat{H}\rangle \in W_{e,\tau} \wedge \hat{H} \subseteq L_\alpha - A^{(0)}_{\sigma}$. We then say that $f^{\sigma}(x) \simeq \hat{y}$ and $\hat{H}$ is the negative neighborhood of this computation.

Further we fix a Σ_2 L_α function g such that $\text{dom}\, g = \omega$ and g maps ω 1-1 onto α. We have in this case a very nice approximation $g^{\cdot}(\cdot)$ to g where $\text{dom}\, g^{\cdot}(\cdot) = \alpha \times \omega$ and $\forall n < \omega\, \exists \sigma\, \forall m \leq n\, \forall \tau \geq \sigma\, (g^{\tau}(m) \simeq g(m))$. We require further that $g^{\sigma}(\cdot)$ is 1-1 for every σ and that $g(0) \simeq g^{\sigma}(0) \simeq 0$ for all σ.

Analogously as in Soare [23] we define functions l and r relative to fixed enumerations $(C_\sigma)_{\sigma<\alpha}$ and $(A_\sigma)_{\sigma<\alpha}$.

For $e, \sigma \in \alpha$ choose $l(e,\sigma) \leq rcf\ D$ maximal such that for all $x < l(e,\sigma)$ the following holds :
There is a stage $\tau \leq \sigma$ such that $f^{\tau}(x)\downarrow$ and the negative neighborhood $\hat{H}$ of this computation satisfies $\hat{H} \subseteq L_\alpha - A_\sigma$ and at stage τ there exists a computation of "$C \leq^{e}_{\alpha} A$" for

"$K_{x,\tau} := f^{\tau}(x) - C_{\tau} \subseteq L_{\alpha} - C$" with negative neighborhood H and H satisfies $H \subseteq L_{\alpha} - A_{\sigma}$ (we then write $\tilde{\tau}$ for the minimal such $\tau \leq \sigma$ and $\tilde{H}$ for the minimal such H).

For $u(e,x,\sigma) := \mu\gamma(\hat{H} \subseteq \gamma \wedge \tilde{H} \subseteq \gamma)$ we then demand in the case that σ is a successor stage that no $y < u(e,x,\sigma)$ was put into A at stage $\sigma-1$.

Finally we demand (for any σ) that $C_{\tilde{\tau}} \cap f^{\sigma}(x) = C_{\sigma} \cap f^{\sigma}(x)$.

If we then have for this $l(e,\sigma)$ that $l(e,\sigma) < \text{rcf } D$ and for $x = l(e,\sigma)$ all the conditions in the definition are satisfied except the last one (i.e. $C_{\tilde{\tau}} \cap f^{\sigma}(x) = C_{\sigma} \cap f^{\sigma}(x)$) we say that e is inactive at stage σ and define

$r(e,\sigma) := \sup\{u(e,x,\sigma) \mid x \leq l(e,\sigma)\}$.

Otherwise we define

$r(e,\sigma) := \sup\{u(e,x,\sigma) \mid x < l(e,\sigma)\}$.

It is convenient to choose the universal enumeration $(W_e)_{e<\alpha}$ in such a way that $W_0 = \emptyset$ so that we have $l(0,\sigma) = r(0,\sigma) = 0$ for all σ.

<u>Construction :</u>

At stage σ we consider every $\langle\beta,\gamma\rangle \in B_{\sigma+1}$ such that $\beta \in \text{Rg } g^{\sigma}(\cdot)$. If $\langle\beta,\gamma\rangle$ is not already an element of A_{σ} we put $\langle\beta,\gamma\rangle$ into A at stage σ if

$\langle\beta,\gamma\rangle \geq r(g^{\sigma}(m),\sigma)$ for all $m \leq n$ where $g^{\sigma}(n) \simeq \beta$.

End of construction.

The claims of Theorem 1 follow as in case i) from the following Lemma .

<u>Lemma 6 :</u> For every $n \in \omega$ we have

a) $A^{(g(n))} =^* B^{(g(n))}$ and

b) $\neg C \leq_\alpha^{g(n)} A$.

<u>Proof:</u> Induction on n . a) and b) are trivial for $n = 0$ since $W_0 = \emptyset$ and for all σ $g^\sigma(0) \simeq 0$. Assume for the following that $n > 0$.

a) We get from the properties of B and the induction hypothesis that $A^{(<n)} := \bigcup \{A^{(e)} \mid g(e) < n\}$ is regular. Choose σ_0 such that $\forall \sigma \geq \sigma_0 \ \forall m \leq n \ (g^\sigma(m) \simeq g(m))$ and define

$T_n := \{\sigma > \sigma_0 \mid \sigma$ is a successor stage and an element y is put into $A^{(<n)}$ at stage $\sigma - 1$ such that $y \cap A^{(<n)} = y \cap A_\sigma^{(<n)}\}$.

Define $I := \{m \leq n \mid \exists \sigma \in T_n \ (g(m)$ is inactive at $\sigma)\}$.

Then there is a stage $\sigma_1 \geq \sigma_0$ such that $\forall \sigma \geq \sigma_1 \ \forall m \in I \ (r(g(m),\sigma) = r(g(m),\sigma_1))$.

Take further any $m \in (n+1) - I$ and assume that $\sup\{r(g(m),\sigma) \mid \sigma \in T_n\} = \alpha$. Then we have $\sup\{l(g(m),\sigma) \mid \sigma \in T_n\} = \mathrm{rcf}\ D$ (by the definition of rcf D) which implies the contradiction $C \leq_\alpha A^{(<n)} \leq_\alpha D$. Thus we have shown that $\sup\{r(g(m),\sigma) \mid m \leq n \wedge \sigma \in T_n\} < \alpha$ which is used for the proof of $A^{(g(n))} =^* B^{(g(n))}$ as usual.

b) $C \leq_\alpha^{g(n)} A$ implies that $\sup\{l(g(n),\sigma) \mid \sigma \in T_n\} = \mathrm{rcf}\ D$ which is absurd according to the preceding.

The proof of Theorem 1 is now finished. We have proved Theorem 1 in order to get the following corollary :

<u>Corollary :</u> Assume that $\sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha$. Then there exist incomplete high α-r.e. degrees.

Proof of the corollary: The case $\alpha = \sigma 2cf\alpha$ is proved in Shore [20] . For the other admissible α there exist incomplete non-hyperregular α-r.e. sets D if $\sigma 2cf\alpha \geq \sigma 2p\alpha$ according to Shore [19] (see also [11] for another proof of this fact). Apply Theorem 1 to this set D and an α-r.e. set $C \in 0'$.

§2. The degree $0^{3/2}$

For those α where incomplete non-hyperregular α-r.e. degrees exist there exists a distinguished α-degree between $0'$ and $0''$ for which we write $0^{3/2}$. We will show in the following and in [11] that there is a close connection between $0^{3/2}$ and the jump of non-hyperregular α-r.e. degrees.

Lemma 7 : Assume α is such that incomplete non-hyperregular α-r.e. degrees exist. Then there is an α-degree $0^{3/2}$ such that

a) $0' <_\alpha 0^{3/2} <_\alpha 0''$

b) $0^{3/2}$ is the greatest $\Delta_2\, L_\alpha$ degree (i.e. $0^{3/2}$ contains a $\Delta_2\, L_\alpha$ set and $D \leq_\alpha 0^{3/2}$ for every $\Delta_2 L_\alpha$ set D)

c) $0^{3/2}$ is the greatest tame-$\Sigma_2\, L_\alpha$ degree (i.e. $0^{3/2}$ contains a set S such that $\{K \in L_\alpha \mid K \subseteq S\}$ is $\Sigma_2\, L_\alpha$ and we have $D \leq_\alpha 0^{3/2}$ for every set D with this property)

d) $U_2^{L_\alpha} \leq_{w\alpha} \underset{\sim}{a} \leftrightarrow 0^{3/2} \leq_\alpha \underset{\sim}{a}$ for the set $U_2^{L_\alpha} \in 0''$ and any $\underset{\sim}{a}$.

Remark: If α is Σ_2 admissible then $0'$ is the greatest $\Delta_2\, L_\alpha$ degree and $0''$ is the greatest tame Σ_2 L degree. Thus for the α of the Lemma they meet together in the middle, one coming from below, the other coming from above.

Proof: $\mathfrak{B} := \langle L_\alpha, C\rangle$ with $C \in 0'$ regular and α-r.e. is inadmissible. A set $S \subseteq L_\alpha$ is $\Delta_2 L_\alpha$ (tame-$\Sigma_2\, L_\alpha$) if and only if S is $\Delta_1 \mathfrak{B}$ (tame-$\Sigma_1 \mathfrak{B}$). Friedman [3] observed that for inadmissible β a greatest $\Delta_1\, L_\beta$ β-degree exists which lies strictly between 0 and $0'$ and which is an upper bound for the tame-$\Sigma_1\, L_\beta$ degrees. This result can't be generalized to all inadmissible structures $\langle L_\beta, D\rangle$ even if D is regular over L_β: The structure $\mathfrak{B} = \langle L_{\aleph_\omega^L}, C\rangle$ with $C \in 0'$ $\aleph_\omega^L$-r.e. and regular is inadmissible (we have $\omega = \sigma 1\mathrm{cf}^{\mathfrak{B}} \aleph_\omega^L < \sigma 1\mathrm{p}^{\mathfrak{B}}\, \aleph_\omega^L = \aleph_\omega^L$) but $0'$ is the greatest $\Delta_1 \mathfrak{B}$ degree. But Friedman's argument works as well for those inadmissible structures $\mathfrak{B} = \langle L_\beta, B\rangle$ where $\sigma 1\mathrm{p}^{\mathfrak{B}}\beta < \beta$. According to Lemma 1 we have $\sigma 2\mathrm{p}\,\alpha < \alpha$ for those α where incomplete non-hyperregular α-r.e. degrees exist. Since we have $\sigma 1\mathrm{p}^{\mathfrak{B}}\alpha = \sigma 2\mathrm{p}\,\alpha$ for the considered structure $\mathfrak{B} = \langle L_\alpha, C\rangle$ there is no problem with the additional assumption in this case.

Take a $\Delta_1 \mathfrak{B}$ set $M \subseteq \alpha$ out of the greatest $\Delta_1 \mathfrak{B}$ $\mathfrak{B}$-degree $\underset{\sim}{r}$ and define $0^{3/2}$ to be the α-degree of the $\Delta_2\, L_\alpha$ set $C \vee M := \{2x \mid x \in C\} \cup \{2x+1 \mid x \in M\}$. Then we have for every set $S \subseteq L_\alpha$ that S is (weakly) $\mathfrak{B}$-recursive in M if and only if S is (weakly) α-recursive in $C \vee M$. Therefore we can prove a) and b) for the so defined α-degree $0^{3/2}$ by using the corresponding properties of the $\mathfrak{B}$-degree $\underset{\sim}{r}$.

In order to prove c) it remains to show that $\underset{\sim}{r}$ contains a tame-$\Sigma_1 \mathfrak{B}$ set. In the case $\sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha$ this follows from Theorem 1 in [9]. If we have $\sigma 2\mathrm{cf}\,\alpha < \sigma 2\mathrm{p}\,\alpha$ then $\mathfrak{B}$ is strongly inadmissible and tame-$\Sigma_1 \mathfrak{B}$ sets which are not of degree 0 may or may not exist for these $\mathfrak{B}$, depending on the fine structure of $\mathfrak{B}$ as it is shown in §2 of [9]. However in our situation where incomplete non-hyperregular α-r.e. degrees exist we have an α-cardinal $\kappa \geq \sigma 2\mathrm{p}\,\alpha$ such that $\sigma 2\mathrm{cf}^{L_\alpha}\kappa = \sigma 2\mathrm{cf}\,\alpha$

according to Lemma 1 . Therefore we can apply the construction of Lemma 5 in [9] and get a tame- $\Sigma_1\mathcal{L}$ set of degree $\underline{r}$.

Property d) follows from Theorem 2 in [9] .

<u>Remark :</u> The greatest Δ_2 L_α and the greatest tame-Σ_2 L_α degree can be determined for the other admissible α as well. The results might be useful for the study of Σ_2 L_α degrees.

For α with $\sigma 2cf\alpha < \sigma 2p\alpha = \alpha$ we have that the greatest Δ_2 L_α degree is equal to $0''$ and the greatest tame-Σ_2 L_α degree is equal to $0'$ (thus these two degrees have switched their places compared with Σ_2 admissible α).

For the other α with the property that $0'$ is the only non-hyperregular α-r.e. degree we have that $\sigma 2cf\alpha < \sigma 2p\alpha < \alpha$ and in this case there is a greatest Δ_2 L degree strictly between $0'$ and $0''$ whereas the greatest tame-Σ_2 degree is either equal to the greatest Δ_2 degree (if $\sigma 2cf^{L_\alpha}(\sigma 2p\alpha) = \sigma 2cf\alpha$) or is equal to $0'$ (otherwise) as one can see by using Lemma 1 and arguments of §2 in [9] .

For all α which are not Σ_2 admissible we have that the greatest Δ_2 L_α degree $\underline{r}$ has the property that $U_2^{L_\alpha} \leq_{w\alpha} \underline{a} \leftrightarrow r \leq_\alpha \underline{a}$ for every α-degree $\underline{a}$.

The following rather technical Lemma will be the heart of the proof of Theorem 2 . It generalizes an observation of Shore (Lemma 3.3 in [18]) which also has important applications in β-recursion theory (see Lemma 3 , §2 in [9]).

Lemma 8 : Consider a structure $\mathcal{L} = \langle L_\beta, B\rangle$ and a limit ordinal $\lambda \leq \beta$ such that $\sigma 1cf^{\mathcal{L}}\beta < \rho_{1,\beta}^{\mathcal{L}}$ and $\sigma 1cf^{\mathcal{L}}\lambda < \rho_{1,\beta}^{\mathcal{L}}$ (see §0. for definitions).

If $D \subseteq L_\lambda$ is regular over L_λ and $\{K \in L_\lambda | K \subseteq D\}$ is $\Sigma_1 \mathcal{L}$ then $\{K \in L_\lambda | K \subseteq L_\lambda - D\}$ is $\Sigma_1 \mathcal{L}$ as well.

Proof: The same trick as in Shore [18] is used. Fix a $\Sigma_1 \mathcal{L}$ definition Ψ of the set $\{K \in L_\lambda | K \subseteq D\}$, a cofinal $\Sigma_1 \mathcal{L}$ function $p : \sigma 1cf^{\mathcal{L}}\lambda \to \lambda$ and a cofinal $\Sigma_1 \mathcal{L}$ function $q : \sigma 1cf^{\mathcal{L}}\beta \to \beta$. Define a $\Pi_1 \mathcal{L}$ set $M \subseteq \sigma 1cf^{\mathcal{L}}\lambda \times \sigma 1cf^{\mathcal{L}}\beta$ by

$$\langle\gamma,\delta\rangle \in M \;:\leftrightarrow\; \forall x \in L_{p(\gamma)}\ (x \in D \to \langle L_{q(\delta)}, L_{q(\delta)} \cap B\rangle \models [\exists K\ (x \in K \wedge \Psi(K))]).$$

Then we have in fact $M \in L_\beta$ and thus get a $\Sigma_1 \mathcal{L}$ definition of $\{K \in L_\lambda | K \subseteq L_\lambda - D\}$ as follows :

$$K \in L_\lambda \wedge K \subseteq L_\lambda - D \leftrightarrow \exists \gamma\,\delta(\langle\gamma,\delta\rangle \in M \wedge K \subseteq L_{p(\gamma)} \wedge K \in L_{q(\delta)} \wedge \langle L_{q(\delta)}, L_{q(\delta)} \cap B\rangle \models [\forall x \in K\ \neg\exists K'(x \in K' \wedge \Psi(K'))]).$$

Theorem 2 : Assume that $\sigma 2cf\alpha < \sigma 2p\alpha$ and $\underset{\sim}{a}$ is an incomplete α-r.e. degree. Then we have

a) $\underset{\sim}{a}' = 0'$ if $\underset{\sim}{a}$ is hyperregular (Shore) and

b) $\underset{\sim}{a}' = 0^{3/2}$ if $\underset{\sim}{a}$ is non-hyperregular .

Proof : a) is contained in Shore [20] . It follows immediately from Lemma 8 : Choose $\mathcal{L} := \langle L_\alpha, C\rangle$ with $C \in 0'$ α-r.e. and regular, $\lambda := \alpha$, $D \in \underset{\sim}{a}'$ regular and $\Sigma_1 \langle L_\alpha, A\rangle$ where $A \in \underset{\sim}{a}$ is α-r.e. and regular .

b) Assume that $A \in \underset{\sim}{a}$ is α-r.e., incomplete, regular and non-hyperregular. Then we have $\alpha > \sigma 1cf^{\langle L_\alpha, A\rangle}\alpha \geq \sigma 1p^{\langle L_\alpha, A\rangle}\alpha$ according to Shore [18] (this fact follows immediately from Lemma 8). For $\kappa := \sigma 1cf^{\langle L_\alpha, A\rangle}\alpha$ we can find a $\Sigma_1\langle L_\alpha, A\rangle$ function g

which maps α 1-1 onto τ. Take any set S which is defined by a Σ_2 formula $\exists y \forall z \phi(x,y,z)$ over L_α. Then we have

$x \notin S \leftrightarrow \forall y \exists z \neg\phi(x,y,z) \leftrightarrow \forall y \in \tau \exists \tilde{y} \exists z \, (g(\tilde{y}) = y \wedge$
$\neg\phi(x,\tilde{y},z)\,) \leftrightarrow \{e\} \times \tau \times \{x\} \subseteq A'$

for some fixed index e. This implies $S \leq_{w\alpha} A'$ (it is this fact which is actually proved in Theorem 2.3. in [20]). We get then $0^{3/2} \leq_\alpha A'$ from Lemma 7 d).

In order to get $0^{3/2} =_\alpha A'$ we show that A' is $\Delta_2\ L_\alpha$ (this implies $A' \leq_\alpha 0^{3/2}$ by Lemma 7 b). Since A' is obviously $\Sigma_2\ L_\alpha$ it is enough to show that A' is $\Pi_2\ L_\alpha$. We do this by showing that $\tilde{A} := f[A']$ is $\Pi_2\ L_\alpha$ where $f : \alpha \to \sigma 1p^{\langle L_\alpha, A\rangle}\alpha$ is a 1-1 $\Sigma_1 \langle L_\alpha, A\rangle$ map. We apply Lemma 8 to the structure $\mathfrak{L} := \langle L_\alpha, C\rangle$ with $C \in 0'$ α-r.e. and regular, $\lambda := \sigma 1p^{\langle L_\alpha, A\rangle}\alpha$ and $D := \tilde{A}$. The assumptions of the Lemma are all satisfied in this situation :

We have $\sigma 1cf^{\mathfrak{L}}\alpha = \sigma 2cf\alpha < \sigma 2p\alpha = \rho_{1,\alpha}^{\mathfrak{L}}$, $\sigma 1cf^{\mathfrak{L}}\lambda \leq \sigma 2cf\alpha < \rho_{1,\alpha}^{\mathfrak{L}}$ (take a cofinal $\Sigma_2\ L_\alpha$ function $q : \sigma 2cf\alpha \to \alpha$; $f \circ q$ is then cofinal in λ because according to Shore [18] we have $\rho_{1,\alpha}^{\langle L_\alpha, A\rangle} = \sigma 1p^{\langle L_\alpha, A\rangle}\alpha \leq \sigma 1cf^{\langle L_\alpha, A\rangle}\alpha$, therefore $f^{-1}[Rg\ f \cap \gamma]$ is bounded for every $\gamma < \lambda$),
$\tilde{A}$ is regular over L_λ (because $\tilde{A}$ is $\Sigma_1 \langle L_\alpha, A\rangle$) and $\{K \in L_\lambda \mid K \subseteq \tilde{A}\}$ is $\Sigma_1 \langle L_\alpha, A\rangle$ (since $\lambda \leq \sigma 1cf^{\langle L_\alpha, A\rangle}\alpha$).

Therefore $\{K \in L_\lambda \mid K \subseteq L_\lambda - \tilde{A}\}$ is $\Sigma_1 \mathfrak{L}$ according to Lemma 8 which implies that $\tilde{A}$ is $\Pi_2\ L_\alpha$.

§3. Summary

Two factors determine the results about the jump of α-r.e. degrees : the relative size of $\sigma 2cf\alpha$ and $\sigma 2p\alpha$ and the existence of incomplete non-hyperregular α-r.e. degrees.

Therefore we distinguish four different types of admissible ordinals α :

(1) $\sigma 2cf\alpha \geq \sigma 2p\alpha$ and there exist no incomplete non-hyperregular α-r.e. degrees
(these are exactly those α which are Σ_2 admissible)

(2) $\sigma 2cf\alpha \geq \sigma 2p\alpha$ and there exist incomplete non-hyperregular α-r.e. degrees
(these are exactly those α which satisfy $\alpha > \sigma 2cf\alpha \geq \sigma 2p\alpha$)

(3) $\sigma 2cf\alpha < \sigma 2p\alpha$ and there exist incomplete non-hyperregular α-r.e. degrees

(4) $\sigma 2cf\alpha < \sigma 2p\alpha$ and there exist no incomplete non-hyperregular α-r.e. degrees .

For the types (2) and (3) there exists the distinguished degree $0^{3/2}$ between $0'$ and $0''$ with the properties that have been described in Lemma 7 .

For α of type (4) we have $\underset{\sim}{a}' = 0'$ for every incomplete α-r.e. degree $\underset{\sim}{a}$ (Shore [20]).

For α of type (3) we have for incomplete α-r.e. degrees $\underset{\sim}{a}$ that $\underset{\sim}{a}' = 0'$ if $\underset{\sim}{a}$ is hyperregular respectively $\underset{\sim}{a}' = 0^{3/2}$ if $\underset{\sim}{a}$ is non-hyperregular according to Theorem 2 .

For α of type (1) and (2) there exist incomplete α-r.e. degrees $\underset{\sim}{a}$ such that $\underset{\sim}{a}' = 0''$ according to §1. (see Shore [20] for type (1)).

In particular we have thus shown the following :

Corollary: Assume that α is admissible. Then there exist high incomplete α-r.e. degrees if and only if $\sigma 2\mathrm{cf}\,\alpha \geq \sigma 2\mathrm{p}\,\alpha$.

We will continue the study of type (1) and (2) in [11]. It turns out that (2) is the most interesting type as far as results about the jump of α-r.e. degrees are concerned.

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