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INADMISSIBILITY, TAME R.E. SETS AND THE ADMISSIBLE COLLAPSE*

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There are some good reasons for studying recursion theory on inadmissible structures. First the study of degrees in α-recursion theory (α is always admissible in this paper) leads naturally to questions about inadmissible structures. A typical example is the minimal α-degree problem, where one wants to construct a minimal α-degree which is recursive in $0'$ (a minimal α-degree cannot be α-r.e.). The natural framework for this construction is the structure $\langle \mathrm{L}_\alpha, C\rangle$ where C is a regular complete Σ_1 set. This structure is inadmissible if α is not Σ_2-admissible (see [5], [10]).

Further the study of degrees in recursion in higher types is closely connected to degree theory on inadmissible structures: for a normal functional $F^{n+2}(n \geq 1)$ the crucial structure $\langle M_{\kappa_n}(F), F\rangle$, which contains all computations in F and a type n object, is inadmissible (see [3], [8]).

A third reason is the conjecture that "Σ_1-admissibility is a crude global hypothesis which obscures the finer points of recursion theory" (Sacks [8]). This conjecture became very convincing because Sy Friedman [2] solved Post's problem for many β which are not admissible. So it seems to be the case that recursion is really not that important in order to prove the basic theorems of recursion theory (the recursion scheme fails in general if the considered ordinal is not admissible). The program is then, to study recursion theory on every limit ordinal β. For this program the results of Jensen about the fine structure of the constructible hierarchy turned out to be of crucial importance. In fact β-recursion theory can be considered to be that part of the fine structure theory of L which deals with questions that are inspired by recursion theory.

Finally, inadmissible recursion theory is interesting from the conceptual point of view. Several definitions which are equivalent in admissible structures define different classes in inadmissible recursion theory. Further, inadmissible structures are a good field for studying those effects which are potentially contained in the basic notions of recursion theory but which can't be studied in admissible structures because they are too "special".

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Inadmissible recursion theory was first considered by Sy Friedman [2][1]. He introduced several definitions of "recursively enumerable" which are equivalent in admissible structures but give rise to different classes in inadmissible structures. We follow Friedman in taking the weakest one (Σ_1-definability) as the definition for "r.e.". The classes which are defined by the stronger definitions (tame r.e., strongly r.e.) are interesting because their elements have more of the properties of an r.e. set in an admissible structure. A characterization of these smaller classes makes it possible to get an idea of those objects which are really new (e.g. sets which are r.e. but not tame r.e.). We would like to compare this situation with the step from ordinary recursion theory to admissible recursion theory. It turned out there, that the study of those α-r.e. sets which are regular respectively hyperregular (properties which every r.e. set in ordinary recursion theory has) was a good way to find those effects which are new in admissible recursion theory.

Results and methods of this paper show that it makes sense to divide inadmissible structures into weakly and strongly inadmissible structures. The Σ_1-cofinality of β, Σ_1-cf β (which is defined by considering only those functions cofinal in β which are β-recursive) is a good measure of the remaining "admissibility" of an inadmissible β. We call β weakly inadmissible, if Σ_1-cf β is large enough so that one can project β β-recursively into this ordinal, we call β strongly inadmissible otherwise.

As the main tool for the study of weakly inadmissible structures we introduce in Section 1 the "admissible collapse". This technique makes it possible to reduce many questions about weakly inadmissible structures to questions about admissible structures.

Concerning the investigation of tame r.e. and strongly r.e. sets in strongly inadmissible β it turns out that it is useful to consider "tame projections", a concept which is introduced in Section 2.

In Section 3 we are going to speculate about the interpretation of some definitions in recursion theory.

The paper is largely self contained but some basic knowledge about constructibility is useful (see Devlin [1] for details).

0. Preliminaries and a remark about gaps in the constructible hierarchy

We follow Friedman in using Jensen's version (J_γ) of the constructible hierarchy, but if the reader doesn't like this hierarchy he may always read $L_{\omega\cdot\gamma}$ for J_γ and L_β for S_β in the following (except for the remark about gaps). More details about the (J_γ) hierarchy can be found in [1].

Every rudimentary function (pairing, etc.) can be obtained as the composition

[1] [2] will be contained in Friedman's forthcoming papers "β-Recursion Theory", "Post's Problem Without Admissibility" and "Forcing in β-Recursion Theory".

of special rudimentary functions $F_0, \dots, F_8$. Then the function

$$\mathcal{S}(U) := (U \cup \{U\}) \cup \left(\bigcup_{i=0}^{8} F_i''(U \cup \{U\}) \right)$$

is a rudimentary function as well. Define the hierarchy $\langle S_\gamma \mid \gamma \in \mathrm{On} \rangle$ by $S_0 = \emptyset$, $S_{\gamma+1} = \mathcal{S}(S_\gamma)$ and $S_\lambda = \bigcup_{\gamma<\lambda} S_\gamma$.

Define for any transitive set U: rud(U) = the smallest $X \supseteq U \cup \{U\}$ such that X is closed under rudimentary functions. If we define then $J_\gamma = S_{\omega \cdot \gamma}$ for every γ, we get the Jensen hierarchy which has the following properties: $J_0 = \emptyset$, $J_{\gamma+1} =$ rud (J_γ), $J_\lambda = \bigcup_{\gamma<\lambda} J_\gamma$. This hierarchy is closely related to the (L_γ) hierarchy. We have $J_0 = \mathrm{L}_0 = \emptyset$ and for all γ $\mathrm{L}_{\omega+\gamma} = V_{\omega+\gamma} \cap J_{1+\gamma}$, hence $J_\gamma = \mathrm{L}_\gamma$ iff $\omega \cdot \gamma = \gamma$ (for example if γ is admissible). Every S_γ is transitive, we have $\gamma < \delta \to S_\gamma \subseteq S_\delta$ and rank $(S_{\omega \cdot \gamma}) = \mathrm{On} \cap S_{\omega \cdot \gamma} = \omega \cdot \gamma$. Further $\langle S_\nu \mid \nu < \omega \cdot \gamma \rangle$ is uniformly $\Sigma_1 S_{\omega \cdot \gamma}$ (i.e. uniformly definable over $S_{\omega \cdot \gamma}$ by a Σ_1-formula).

For the rest of the paper β is always a limit ordinal.

A total function from β onto S_β, $\eta \mapsto K_\eta$ is Δ_1-definable over every S_β (though not uniformly). A total 1-1 onto pairing function $\beta \times \beta \to \beta$ is Δ_1-definable over every S_β as well. We are going to reserve the letter K for β-finite sets, i.e. elements of S_β.

Let $B \subseteq S_\beta$ be regular over S_β, i.e. $\forall \sigma < \beta \, (B \cap S_\sigma \in S_\beta)$. We always write $\mathfrak{B}$ for the structure $\langle S_\beta, B \rangle$ (or more exactly, $\langle S_\beta, \in, B \rangle$).

A set $A \subseteq S_\beta$ is defined to be $\mathfrak{B}$-recursively enumerable ($\mathfrak{B}$-r.e.) iff A is Σ_1-definable over $\mathfrak{B}$. $A \subseteq S_\beta$ is $\mathfrak{B}$-recursive iff A and $S_\beta - A$ are $\mathfrak{B}$-r.e. A (partial) function $f: S_\beta \to S_\beta$ is called $\mathfrak{B}$-recursive iff the graph of f is $\mathfrak{B}$-r.e. (a function $g: M \to N$ is always totally defined on M if we don't say that g is partial). We write r.e., recursive etc., if it is clear which $\mathfrak{B}$ we mean.

Σ_1-cf $\mathfrak{B}$, the Σ_1-cofinality of $\mathfrak{B}$, is the least ordinal $\gamma \leq \beta$ such that a $\mathfrak{B}$-recursive cofinal function $f: \gamma \to \beta$ exists. $\mathfrak{B}^*$, the Σ_1-projectum of $\mathfrak{B}$, is the least $\gamma \leq \beta$ such that a $\mathfrak{B}$-recursive 1-1 function $f: \beta \to \gamma$ exists. $\mathfrak{B}$ is called admissible, if Σ_1-cf $\mathfrak{B}$ $= \beta$. If $\mathfrak{B}$ is inadmissible and Σ_1-cf $\mathfrak{B} \geq \mathfrak{B}^*$ we call $\mathfrak{B}$ weakly inadmissible, if Σ_1-cf $\mathfrak{B} < \mathfrak{B}^*$ we call $\mathfrak{B}$ strongly inadmissible.

Both Σ_1-cf $\mathfrak{B}$ and $\mathfrak{B}^*$ are β-cardinals, i.e. $\mathfrak{B} \models (\Sigma_1\text{-cf } \mathfrak{B}$ is a cardinal) and $\mathfrak{B} \models (\mathfrak{B}^*$ is a cardinal) unless Σ_1-cf $\mathfrak{B}$ or $\mathfrak{B}^*$ equal β. The following property of β-cardinals is proved in a way similar to $\mathrm{L} \models \mathrm{GCH}$. If ρ is a β-cardinal, $K \in S_\beta$ and $K \subseteq S_\sigma$ for some $\sigma < \rho$ then $K \in S_\rho$. (The usual argument is dubious in the case where β has the form $\gamma + \omega$. But in this case we can apply the uniformization theorem to the set S_γ.) Friedman [2] proved that every β-cardinal $\rho > \omega$ is Σ_1-stable, i.e. $S_\rho <_{\Sigma_1} S_\beta$ (S_ρ is a Σ_1-elementary substructure of S_β). This shows that for all inadmissible sets S_β a largest β-cardinal less than β exists (this result is in general not true for inadmissible structures $\mathfrak{B}$). An easy stability argument shows that $\beta^* < \beta$ for all inadmissible sets S_β which implies that every β-cardinal is admissible.

Reducibility relations for sets $A, D \subseteq S_\beta$ are defined as follows:

$A \leqslant_{\mathfrak{B}} D \leftrightarrow$ there exist $\mathfrak{B}$-r.e. sets W_e, $W_{e'}$ such that for all $K \in S_\beta$

$$K \subseteq A \leftrightarrow \exists H_1, H_2 \in S_\beta\ (\langle K, H_1, H_2\rangle \in W_e \wedge H_1 \subseteq D \wedge H_2 \subseteq S_\beta - D),$$

$$K \subseteq S_\beta - A \leftrightarrow \exists H_1, H_2 \in S_\beta\ (\langle K, H_1, H_2\rangle \in W_{e'} \wedge H_1 \subseteq D \wedge H_2 \subseteq S_\beta - D).$$

$A \leqslant_{w\mathfrak{B}} D \leftrightarrow$ there exist $\mathfrak{B}$-r.e. sets W_e, $W_{e'}$ such that for all $x \in S_\beta$

$$x \in A \leftrightarrow \exists H_1, H_2 \in S_\beta\ (\langle x, H_1, H_2\rangle \in W_e \wedge H_1 \subseteq D \wedge H_2 \subseteq S_\beta - D),$$

$$x \in S_\beta - A \leftrightarrow \exists H_1, H_2 \in S_\beta\ (\langle x, H_1, H_2\rangle \in W_{e'} \wedge H_1 \subseteq D \wedge H_2 \subseteq S_\beta - D).$$

One usually reads "$A \leqslant_{\mathfrak{B}} D$" as "$A$ is recursive in D" and "$A \leqslant_{w\mathfrak{B}} D$" as "$A$ is weakly recursive in D" (see Section 3 below for some problems concerning this interpretation).

$\mathfrak{B}$-degrees are the equivalence classes on $\mathfrak{p}(S_\beta)$ which are generated by the relation "$=_{\mathfrak{B}}$". We always write $0_{\mathfrak{B}}$ for the degree of the empty set and $0'_{\mathfrak{B}}$ for the degree of a universal $\mathfrak{B}$-r.e. set. We say that a degree has certain properties $P_1, \ldots, P_n$ iff there exists an element of the degree which has all the properties $P_1, \ldots, P_n$.

Remark 1. Observe that the notions of β-recursion theory are suitable for proving very easily most of the results about the length of gaps (see [7]) for the (J_γ)-hierarchy, using facts from above and Jensen's uniformization theorem.

As an example we consider *Theorem* 4.4 *from Marek–Srebrny* [7]. *Let* $\rho, \gamma \in \omega_1^L$ *be given such that* $\gamma > \rho$. *Let* δ *be the first ordinal greater than* γ *which starts a gap of length* $\geqslant \rho$. *Then this gap has exactly length* ρ. (*By definition* δ *starts a gap of length* ρ *iff*

$$\forall \sigma < \delta\ ((J_\delta - J_\sigma) \cap \mathfrak{p}(\omega) \neq \emptyset \wedge (J_{\delta+\rho} - J_\delta) \cap \mathfrak{p}(\omega) = \emptyset \wedge (J_{\delta+\rho+1} - J_\delta) \cap \mathfrak{p}(\omega) \neq \emptyset).)$$

Proof by contradiction. Define $\beta := \omega \cdot (\delta + \rho + 1)$. Let $\kappa \in \beta$ be the β-cardinality of $\omega \cdot (\delta + \rho)$. By our assumption and the Uniformization Theorem we have $\kappa > \omega$ and in fact $\kappa \geqslant \delta$ (since δ *starts* a gap). Consider the following Σ_1 formula Φ:

$$\Phi :\equiv \exists \sigma, \tau\ (\sigma > \gamma \wedge \tau = \sigma + \rho \wedge \forall \sigma' < \sigma\ ((J_\sigma - J_{\sigma'}) \cap \mathfrak{p}(\omega) \neq \emptyset) \wedge (J_\tau - J_\sigma) \cap \mathfrak{p}(\omega) = \emptyset).$$

We have $S_\beta \models \Phi$, therefore by the Σ_1-stability of κ, $S_\kappa \models \Phi$, a contradiction.

Other results about the length of gaps are derived in a similar fashion, using appropriate β and Σ_1 formulae Φ.

Results about partial gaps follow as well. Devlin proved in [7]: if γ is not a gap ordinal, then $\gamma + 1$ is not a Δ_1-gap ordinal (i.e. $(\Delta_1 J_{\gamma+1} - J_{\gamma+1}) \cap \mathfrak{p}(\omega) \neq \emptyset$).

A proof in β-recursion theory goes as follows. Consider S_β with $\beta = \omega \cdot (\gamma + 1)$. β is inadmissible, therefore $\beta^* < \beta$ and of course $\beta^* \leqslant \omega \cdot \gamma$. Further ω is the β-cardinality of $\omega \cdot \gamma$ because γ is not a gap ordinal (Uniformization Theorem), therefore $\beta^* = \omega = \Sigma_1\text{-cf}\,\beta$. Then there exists a 1-1 β-recursive projection from β onto ω (see Section 1) which implies that $\gamma+1$ is not a Δ_1-gap.

1. The admissible collapse of weakly inadmissible structures

It is obvious that the Σ_1-replacement Axiom holds in inadmissible structures $\mathfrak{B}$ for functions which have a β-finite domain of β-cardinality less than $\Sigma_1\text{-cf}\,\mathfrak{B}$. Further $\Sigma_1\text{-cf}\,\mathfrak{B}$ determines, how much "recursion" we have in $\mathfrak{B}$.

Lemma 2. $\Sigma_1\text{-cf}\,\mathfrak{B}+1$ *is the least ordinal* γ, *such that the following scheme of definition by recursion over* γ *fails*:

If $G: S_\beta \times \gamma \to S_\beta$ *is a total* $\mathfrak{B}$*-recursive function then there exists a total* $\mathfrak{B}$*-recursive function* $F: \gamma \to S_\beta$ *such that*

$$\forall \delta < \gamma\,(F \restriction \delta \in S_\beta \wedge F(\delta) = G(F \restriction \delta, \delta)).$$

Proof. Obvious. Observe that one doesn't really need that G is totally defined on $S_\beta \times \gamma$ for $\gamma \leq \Sigma_1\text{-cf}\,\mathfrak{B}$. It is enough to know that $\operatorname{dom} G$ is such that a total function $F: \gamma \to S_\beta$ exists somewhere in V which satisfies the recursion equation for all $\delta < \gamma$.

By Lemma 2 we can assume that the cofinal $\mathfrak{B}$-recursive function $q: \Sigma_1\text{-cf}\,\mathfrak{B} \to \beta$ is in addition strictly increasing and continuous. We are going to write q for such a function in the following and we always write κ for $\Sigma_1\text{-cf}\,\mathfrak{B}$.

Friedman proved that one can always define a $\mathfrak{B}$-recursive projection of β onto $\max(\Sigma_1\text{-cf}\,\mathfrak{B}, \mathfrak{B}^*)$. For the case that $\mathfrak{B}$ is weakly inadmissible one has another proof of this fact. Let $f: \beta \to \mathfrak{B}^*$ be a $\mathfrak{B}$-recursive projection. Define $\tilde{P}: \beta \to \Sigma_1\text{-cf}\,\mathfrak{B}$ by $\tilde{P}(\delta) = \langle \sigma, \tau \rangle \leftrightarrow (\sigma$ is minimal such that $S_{q(\sigma)} \models f(\delta) = \tau)$. Since $\operatorname{rg} \tilde{P}$ is $\Delta_1\mathfrak{B}$ we can define by Lemma 2 a 1-1 onto $\mathfrak{B}$-recursive map g: $\Sigma_1\text{-}cf\,\mathfrak{B} \to \operatorname{rg} \tilde{P}$. Define then $P := g^{-1} \cdot \tilde{P}$.

For the rest of this paragraph we assume that $\mathfrak{B}$ is weakly inadmissible and we fix a $\mathfrak{B}$-recursive projection P of β onto κ. We assume for convenience that P has the property that $\forall x \in \kappa\,(P(x) = 2 \cdot x)$.

The following predicate $\tilde{T} \subseteq \beta^3$ for $\mathfrak{B}$ is defined similarly to Kleene's T-predicate: $\langle x, y, z \rangle_s \in \tilde{T} \leftrightarrow \langle S_x, S_x \cap B \rangle \models \Phi(y, z)$, where $\Phi(y, z)$ is a Σ_1 formula which defines the universal Σ_1 predicate over $\mathfrak{B}$. We always write $\langle \cdot, \cdot, \cdot \rangle_s$ if we want to emphasize that set theoretic pairing is used.

We collapse $\tilde{T}$ to the predicate $T \subseteq \kappa$ by defining $\langle x, y, z \rangle \in T \leftrightarrow \langle P^{-1}(x), P^{-1}(y), P^{-1}(z) \rangle \in \tilde{T}$. T is, as well as $\tilde{T}$, Δ_1 definable over $\mathfrak{B}$. We call the structure $\mathfrak{A} := \langle S_\kappa, T \rangle$ the admissible collapse of $\mathfrak{B}$. T is regular over S_κ, because $T \cap \gamma \in S_\beta$ for every $\gamma < \kappa$ and κ is a β-cardinal.

If $A \subseteq \kappa$ is $\Sigma_1\mathfrak{A}$ then A is $\Sigma_1\mathfrak{B}$. Let Ψ be the Σ_1 formula which defines A over

$\mathfrak{A}$. Then

$$x \in A \leftrightarrow \mathfrak{B} \models \exists \lambda \in \kappa\, \exists K, H\, (K \subseteq T \wedge H \subseteq \kappa - T \wedge K \cup H = \lambda \wedge \langle S_\lambda, K\rangle \models \Psi(x))$$

and by using the definition of κ this can be written as a Σ_1 formula.

On the other hand $A\Sigma_1\mathfrak{B}$ implies $A\Sigma_1\mathfrak{A}$ for $A \subseteq \kappa$ because then for some $e \in \beta$ we have $x \in A \leftrightarrow \mathfrak{A} \models \exists y\, (\langle y, P(e), x\rangle \in T)$. This implies that $\mathfrak{A}$ is admissible and that $\mathfrak{A}$-recursive ($\mathfrak{A}$-r.e.) is the same as $\mathfrak{B}$-recursive ($\mathfrak{B}$-r.e.) as far as subsets of S_κ are concerned.

Remark 3. Harrington defined in [3] (see also [8]) a collapse which used a similarly defined predicate T. His collapse makes it possible to reduce some questions about degrees of type $n+2$ objects which are r.e. in some fixed functional $F^{n+2}(n \geq 1)$ to questions about sets which are Σ_1 definable over an admissible structure $\langle L_\alpha[T], T\rangle$ (T is there not regular over L_α). The main purpose of Harrington's collapse is the elimination of the gaps between subconstructive stages of the computation hierarchy.

Although we have shown so far that we can represent $\mathfrak{B}$-r.e. sets A by $\mathfrak{A}$-r.e. sets in the admissible $\mathfrak{A}$ (consider the projection $P[A]$ of A), this doesn't help much if we want to get information about degrees in $\mathfrak{B}$ by considering degrees in $\mathfrak{A}$. The reason is the following. If $A, B \subseteq \kappa$ and $A \leq_{\mathfrak{A}} B$, this doesn't imply $A \leq_{\mathfrak{B}} B$ because the reduction procedure in $\mathfrak{A}$ doesn't reduce questions $K \subseteq A$, $K \subseteq \kappa - A$ for $K \in S_\beta - S_\kappa$ to questions about B. Analogously $A \leq_{\mathfrak{B}} B$ doesn't imply $A \leq_{\mathfrak{A}} B$ for $A, B \subseteq \kappa$ because the reduction procedure for $A \leq_{\mathfrak{B}} B$ might ask questions $K \subseteq B$, $K \subseteq \kappa - B$ for $K \in S_\beta - S_\kappa$ in order to answer questions $K' \subseteq A$, $K' \subseteq \kappa - A$ for some $K' \in S_\kappa$. We call sets $A \subseteq \kappa$ β-immune, if they are "immune" with respect to those sets $K \subseteq \kappa$ with $K \in S_\beta - S_\kappa$ which cause this trouble.

Definition. $A \subseteq \kappa$ is *β-immune* $\leftrightarrow \forall K \in S_\beta\, (K \subseteq A \vee K \subseteq \kappa - A \to K \in S_\kappa)$.

A β-immune set A has the property that for every $B \subseteq \kappa$

$$A \leq_{\mathfrak{A}} B \to A \leq_{\mathfrak{B}} B$$

and

$$B \leq_{\mathfrak{B}} A \to B \leq_{\mathfrak{A}} A.$$

Especially if B is β-immune as well we get $A \leq_{\mathfrak{A}} B \leftrightarrow A \leq_{\mathfrak{B}} B$.

Although β-immune seems to be a strong requirement, the following construction shows that every degree in $\mathfrak{A}$ contains a β-immune set. The idea is the following. We construct a partial $\mathfrak{A}$-recursive characteristic function χ_M of some (partial) set $M \subseteq \kappa$. We insure that the order type of κ-dom χ_M is κ and that for any set $M' \subseteq \kappa$ the following holds. If $\chi_M \subseteq \chi_{M'}$ ($\chi_{M'}$ is the characteristic function for M') then M' is β-immune. For any set $A \subseteq \kappa$ we can find then a β-immune set

$\tilde{A}$ of the same $\mathfrak{A}$-degree by inserting χ_A into κ-dom χ_M. This defines the characteristic function of some set $\tilde{A}$ with $\chi_M \subseteq \chi_{\tilde{A}}$.

We do this formally by defining a function $f:\kappa \to \kappa$ such that f has only values of the form $\langle 0,0\rangle$, $\langle 0,1\rangle$, $\langle 1,x\rangle$ with $x\in\kappa$. If $f(\sigma)=\langle 0,0\rangle$ or $f(\sigma)=\langle 0,1\rangle$ then $\chi_M(\sigma)$ is defined and has value 0 or 1 respectively (we define $\chi_M(\sigma)=1$ for elements $\sigma\in M$). $f(\sigma)=\langle 1,x\rangle$ simply says that σ is the xth element of κ-dom χ_M.

The function f is constructed as follows. We take a $\mathfrak{B}$-recursive enumeration G of all sets $K\subseteq\kappa$ such that $K\in S_\beta - S_\kappa$ where dom $G=\kappa$. We define by recursion over κ a $\mathfrak{B}$-recursive function $F:\kappa\to S_\beta$ such that for every $\sigma<\kappa$ $F(\sigma)$ is the graph of an initial segment of f which is defined on an ordinal τ where $\sigma\leq\tau<\kappa$. We write $F(<\sigma)$ for $\bigcup_{\rho<\sigma}F(\rho)$. In the recursion step we take

$$\begin{aligned}F(\sigma) &= F(<\sigma)\cup\langle \operatorname{dom} F(<\sigma),\langle 1,\sigma'\rangle\rangle && \text{if}\quad \sigma=3\cdot\sigma',\\ &= F(<\sigma)\cup\{\langle\tau,\langle 0,0\rangle\rangle \mid \operatorname{dom} F(<\sigma)\leq\tau\leq\mu x(x\geq \operatorname{dom} F(<\sigma)\wedge x\in G(\sigma'))\} && \text{if}\quad \sigma=3\cdot\sigma'+1,\\ &= F(<\sigma)\cup\{\langle\tau,\langle 0,1\rangle\rangle \mid \operatorname{dom} F(<\sigma)\leq\tau\leq\mu x(x\geq \operatorname{dom} F(<\sigma)\wedge x\in G(\sigma'))\} && \text{if}\quad \sigma=3\cdot\sigma'+2.\end{aligned}$$

This construction succeeds because dom $F(<\sigma)<\kappa$ for all $\sigma<\kappa$ due to the definition of κ as Σ_1-cf $\mathfrak{B}$.

For any set $A\subseteq\kappa$ we define $\tilde{A}$ by setting

$$\begin{aligned}\chi_{\tilde{A}}(\sigma) &= 0 && \text{if}\quad f(\sigma)=\langle 0,0\rangle,\\ &= 1 && \text{if}\quad f(\sigma)=\langle 0,1\rangle,\\ &= \chi_A(x) && \text{if}\quad f(\sigma)=\langle 1,x\rangle.\end{aligned}$$

It is obvious that $\mathfrak{A}$-recursive functions g, h can be defined such that for any $A\subseteq\kappa$ and $x\in\kappa$, $x\in A\leftrightarrow g(x)\in\tilde{A}$ and, if $A\neq\emptyset$ and $A\neq\kappa$, $x\in\tilde{A}\leftrightarrow h(x)\in A$. This implies that for any $A\subseteq\kappa$, $A=_{\mathfrak{A}}\tilde{A}$. We further have that A is Σ_n $\mathfrak{A}(\Pi_n\mathfrak{A},\Delta_n\mathfrak{A})$ iff $\tilde{A}$ is $\Sigma_n\mathfrak{A}(\Pi_n\mathfrak{A},\Delta_n\mathfrak{A})$ for every $n\geq 1$ and A is regular over S_κ iff $\tilde{A}$ is regular over S_κ.

We are now in the position to determine the relations between some stronger notions of "r.e." in $\mathfrak{B}$. At the same we find out which $\mathfrak{B}$-r.e. degrees are represented by β-immune $\mathfrak{A}$-r.e. sets.

First we recall some definitions from [2]. Let $\mathfrak{B}=\langle S_\beta, B\rangle$ be any structure. $A\subseteq S_\beta$ is tame r.e. (t.r.e.) in $\mathfrak{B}\leftrightarrow\{K\in S_\beta \mid K\subseteq A\}$ is $\mathfrak{B}$-r.e. For $A\subseteq S_\beta$ let $\Delta_0^+(A)$ be the set of those first order formulae with parameters from S_β, which consist of a string of bounded quantifiers followed by a matrix in which (beside the predicate B) the predicate A occurs, but A occurs only positively. We identify these formulae with ordinals in β by some fixed coding.

Definition. A is *strongly r.e.* (*s.r.e.*) in $\mathfrak{B}\leftrightarrow$ the formulae of $\Delta_0^+(A)$ which are true in $\langle\mathfrak{B}, A\rangle$ form a $\mathfrak{B}$-r.e. set.

Definition. A is *n-r.e.*$\leftrightarrow$the true sentences of $\Delta_0^+(A)$ involving $(n-1)$ alternations of quantifiers form a $\mathfrak{B}$-r.e. set.

For later use we further introduce:

Definition. A is *n'-r.e.*$\leftrightarrow$the true sentences of $\Delta_0^+(A)$ involving exactly n quantifiers form a $\mathfrak{B}$-r.e. set.

Observe that for admissible $\mathfrak{B}$ all these notions give the same as $\mathfrak{B}$-r.e. Further for any $\mathfrak{B}$ we have that t.r.e.+regular implies s.r.e., s.r.e. implies n-r.e. and n'-r.e. for all n and $1'$-r.e. implies t.r.e. ([2]).

Theorem 4. *Assume $\mathfrak{B}$ is weakly inadmissible. Then for any degree b in $\mathfrak{B}$ the following are equivalent:*
(1) *b contains a t.r.e. set,*
(2) *b contains a β-immune $\mathfrak{A}$-r.e. set,*
(3) *b contains a t.r.e. regular subset D of β such that every $K\subseteq D$ has an order type less than κ and such that $\forall C\subseteq S_\beta\ (D\leqslant_{w\mathfrak{B}} C\rightarrow D\leqslant_{\mathfrak{B}} C)$,*
(4) *b contains a recursive set,*
(5) *b contains a set which is recursive, t.r.e. and regular.*

Proof. (1)$\rightarrow$(2). Let $A\subseteq S_\beta$ be t.r.e. We first construct a t.r.e set A^* such that

(a) $A=_{\mathfrak{B}}A^*$

and

(b) $\forall C\subseteq S_\beta\ (A^*\leqslant_{w\mathfrak{B}} C\rightarrow A^*\leqslant_{\mathfrak{B}} C)$.

Define $A':=\{\eta\mid K_\eta\cap A\neq\emptyset\}$ and take $A^*:=q[P[A']]$. Then we have

$$K\subseteq S_\beta-A\leftrightarrow\exists\eta\ (K=K_\eta\wedge q(P(\eta))\in S_\beta-A^*)$$

which shows $A\leqslant_{\mathfrak{B}}A^*$ since A is t.r.e. We further have

$$K\subseteq S_\beta-A^*\leftrightarrow\exists\sigma,K',\eta\ (K\subseteq q(\sigma)\wedge K'=\{\tau\in\sigma\mid q(\tau)\in K\}\wedge$$
$$K_\eta=\bigcup\{K_{\eta'}\mid P(\eta')\in K'\}\wedge q(P(\eta))\in S_\beta-A^*).$$

A^* is t.r.e. because A^* is $\mathfrak{B}$-r.e. and every $K\subseteq A^*$ has an order type less than κ. This establishes (b). Since $A^*\leqslant_{w\mathfrak{B}}A$, we have (a).

We define then $C:=P[A^*]$ and take a β-immune set $\tilde{C}$ out of the $\mathfrak{A}$-degree of C as described previously. Then $A^*\leqslant_{w\mathfrak{B}}\tilde{C}$ is obvious and we get $A^*\leqslant_{\mathfrak{B}}\tilde{C}$ by the property of A^*. Since $\tilde{C}$ is β-immune we have to consider only sets $K\in S_\kappa$ for the proof of $\tilde{C}\leqslant_{\mathfrak{B}}A^*$. Since $\tilde{C}$ is reducible to C by the $\mathfrak{A}$-recursive function h (except for trivial cases), we may reduce questions $K\subseteq\tilde{C}$, $K\subseteq S_\beta-\tilde{C}$ to questions $K'\subseteq C$, $K'\subseteq\kappa-C$ with $K'\in S_\kappa$. For these sets K' we have $P^{-1}[K']\in S_\beta$ so that the latter questions can be reduced to $P^{-1}[K']\subseteq A^*$, $P^{-1}[K']\subseteq S_\beta-A^*$. This shows $\tilde{C}\leqslant_{\mathfrak{B}}A^*$.

(2)$\to$(3). For A β-immune and r.e. we take an r.e. set $D' \subseteq \kappa$ such that $A =_{\mathfrak{A}} D'$ and D' is regular over S_κ (apply the Regular Set Theorem in $\mathfrak{A}$). We proceed then to $\tilde{D}'$ which is again of the same $\mathfrak{A}$-degree as A, regular, r.e. and in addition β-immune. Define $D'' := \{\eta \in \kappa \mid K_\eta \cap \tilde{D}' \neq \emptyset\}$, using an $\mathfrak{A}$-recursive map $\eta \to K_\eta$ from κ onto S_κ. Then we take $D := q[D'']$. D is $\mathfrak{B}$-r.e. and because every $K \in S_\beta$ with $K \subseteq D$ has an order type less than κ D is in fact t.r.e. D is regular over S_β because $\tilde{D}'$ and D'' are regular over S_κ. We have further that for any $K \in S_\beta$

$$K \subseteq S_\beta - D \leftrightarrow \exists \sigma, K', \eta\ (q(\sigma) > K \wedge K' = \{\tau \in \sigma \mid q(\tau) \in K\} \wedge \\ \eta \in \kappa \wedge K_\eta = \bigcup \{K_\tau \mid \tau \in K'\} \wedge q(\eta) \in S_\beta - D).$$

$\tilde{D}'$ is t.r.e. and for any $K \in S_\beta$:

$$K \subseteq S_\beta - \tilde{D}' \leftrightarrow \exists \eta \in \kappa\ \exists K' \in S_\kappa\ (K' = K \cap \kappa \wedge K_\eta = K' \wedge q(\eta) \in S_\beta - D),$$

which shows $\tilde{D}' \leqslant_{\mathfrak{B}} D$. We have $D \leqslant_{w\mathfrak{B}} \tilde{D}'$ since

$$x \in S_\beta - D \leftrightarrow \exists \sigma\ (q(\sigma) < x < q(\sigma+1) \vee (x = q(\sigma) \wedge K_\sigma \subseteq S_\beta - \tilde{D}')).$$

Since this implies $D \leqslant_{\mathfrak{B}} \tilde{D}'$ and $A =_{\mathfrak{A}} \tilde{D}'$ implies $A =_{\mathfrak{B}} \tilde{D}'$ the proof of this step is complete.

(3)$\to$(1). Trivial.

(2)$\to$(5). Take a set $A \in b$ according to (2) and let Φ be a Σ_1 formula which defines A over $\mathfrak{A}$. According to the proof of (2)$\to$(3) we may assume that A is regular over S_κ. We define

$$A' := \{\langle \sigma, \eta \rangle_s \mid \sigma \in \kappa \wedge \exists \delta \in \kappa\ (q(\delta) = \eta \wedge \langle S_\sigma, S_\sigma \cap T \rangle \models \Phi(\delta))\}.$$

A' is obviously regular and A' is t.r.e. because we have for any $K \in S_\beta$:

$$K \subseteq A' \leftrightarrow \mathfrak{B} \models \exists K'\ (K' = \{\eta \in \mathrm{On} \mid \exists \sigma \in \kappa\ (\langle \sigma, \eta \rangle \in K)\} \wedge K \subseteq \kappa \times K' \wedge \\ \exists K'' \in S_\kappa\ (K' = q[K''] \wedge K'' \subseteq A \wedge \\ \forall \delta \in K'' \exists \tau \in \kappa\ (\langle S_\tau, S_\tau \cap T \rangle \models [\Phi(\delta)] \wedge \\ \forall \sigma \in \kappa\ (\langle \sigma, q(\delta) \rangle \in K \to \tau \leqslant \sigma)))).$$

Using that q is continuous we get that A' is recursive. We get $A \leqslant_{\mathfrak{B}} A'$ because for any $K \in S_\beta$:

$$K \subseteq S_\beta - A \leftrightarrow \exists K' \in S_\kappa\ (K' = K \cap \kappa \wedge \kappa \times q[K'] \subseteq S_\beta - A').$$

Finally we have $A' \leqslant_{\mathfrak{B}} A$ because for any $K \in S_\beta$:

$$K \subseteq S_\beta - A' \leftrightarrow \mathfrak{B} \models \exists \sigma_0 (K \in S_{q(\sigma_0)} \wedge \exists K'\ (K' = \{\delta \in \sigma_0 \mid \exists \sigma \in \kappa\ (\langle \sigma, q(\delta) \rangle \in K)\} \\ \wedge \exists K_1, K_2 \in S_\kappa\ (K' = K_1 \cup K_2 \wedge K_1 \subseteq A \wedge K_2 \subseteq S_\beta - A \\ \wedge \forall \delta \in K_1\ \exists \tau \in \kappa\ (\langle S_\tau, S_\tau \cap T \rangle \models [\Phi(\delta)] \\ \wedge \forall \tau' < \tau (\langle S_{\tau'}, S_{\tau'}, \cap T \rangle \Vdash \neg \Phi(\delta) \\ \wedge \forall \sigma \in \kappa\ (\langle \sigma, q(\delta) \rangle \in K \to \sigma < \tau))))))$$

(we use here that A is regular over S_κ).

(4)→(1). Let $A \in b$ be recursive. Define $A' \subseteq \{0,1\} \times S_\beta$ by

$$A' := \{\langle 0, K\rangle \mid K \cap A \neq \emptyset\} \cup \{\langle 1, K\rangle \mid K \cap S_\beta - A \neq \emptyset\}$$

and take $D := q[P_U[A']]$ (P_U is a recursive projection of S_β onto κ). Then D is r.e. and $A \leqslant_{\mathfrak{B}} D$ is obvious, because for any $K \in S_\beta$:

$$K \subseteq A \leftrightarrow \langle 1, K\rangle \in S_\beta - A' \leftrightarrow q(P_U(\langle 1, K\rangle)) \in S_\beta - D$$

and

$$K \subseteq S_\beta - A \leftrightarrow \langle 0, K\rangle \in S_\beta - A' \rightarrow q(P_U(\langle 0, K\rangle)) \in S_\beta - D.$$

In addition we have for any $K \in S_\beta$:

$$\begin{aligned} K \subseteq D \leftrightarrow \exists K' \in S_\kappa\ (K = q[K'] \wedge \exists K_1, K_2 \in S_\kappa\ (K_1 = \{\tilde{K} \mid \langle 0, \tilde{K}\rangle \in P_U^{-1}[K']\} \wedge \\ K_2 = \{\tilde{K} \mid \langle 1, \tilde{K}\rangle \in P_U^{-1}[K']\} \wedge K_1 \cup K_2 = P_U^{-1}[K'] \wedge \\ \forall \tilde{K} \in K_1 \exists x\, (x \in \tilde{K} \wedge x \in A) \wedge \forall \tilde{K} \in K_2\, \exists x\, (x \in \tilde{K} \wedge x \in S_\beta - A))). \end{aligned}$$

We can express this by a Σ_1 formula over $\mathfrak{B}$, which shows that D is t.r.e., because K_1 and K_2 have in $\mathfrak{B}$ cardinality less than κ.

Finally $D \leqslant_{\mathfrak{B}} A$ follows from

$$\begin{aligned} K \subseteq S_\beta - D \leftrightarrow \exists \sigma_0\, (K \in S_{q(\sigma_0)} \wedge \exists K'(K' = \{\sigma \in \sigma_0 \mid q(\sigma) \in K\} \wedge \\ \exists K_1, K_2\, (K_1 = \bigcup \{\tilde{K} \mid \langle 0, \tilde{K}\rangle \in P_U^{-1}[K']\} \wedge K_1 \subseteq S_\beta - A \wedge \\ K_2 = \bigcup \{\tilde{K} \mid \langle 1, \tilde{K}\rangle \in P_U^{-1}[K']\} \wedge K_2 \subseteq A))). \end{aligned}$$

Remark 5. All steps in the preceeding proof except (2)→(3) and (2)→(5) are obviously uniform, i.e. we can define for these steps $\mathfrak{B}$-recursive functions $f_{i,j}$ such that $f_{i,j}$ computes for a given index e of a set $W_e \in b$ with property i the index of a set $W_{f_{i,j}(e)} \in b$ with property j. The steps (2)→(3) and (2)→(5) are not obviously uniform but nevertheless uniform, because the Regular Set Theorem for $\mathfrak{A}$ (which is applied for these steps) is now available in an uniform version ([6]).

Remark 6. Theorem 4 contains the following (uniform) Regular Set Theorem for weakly inadmissible structures $\mathfrak{B}$: for every t.r.e. set C we can find (uniformly) a regular t.r.e. set $D \subseteq \beta$ with $C =_{\mathfrak{B}} D$.

A first step toward a regular set theorem in inadmissible recursion theory was taken by Friedman [2]. He proved that for inadmissible β with gc β (:= the greatest β-cardinal) not greater than max (Σ_1-cf β, β^*) the following holds. Every s.r.e. set has the same β-degree as some s.r.e. regular set. The restriction for gc β was not mentioned in [2], but is needed at point d), p. 47. The restriction to s.r.e. sets is essential, because we only know through the regular set theorem for t.r.e. sets above that t.r.e. and s.r.e. degrees are the same in weakly inadmissible $\mathfrak{B}$. A regular set theorem for s.r.e. sets in strongly inadmissible β is of no interest because we have there $A =_\beta \emptyset$ for every s.r.e. set A (see Section 2 below).

Remark 7. There are weakly inadmissible β such that the t.r.e. degrees coincide with the degrees containing regular β-r.e. subsets of β (see Example 18 below).

Let R be the set of those $\mathfrak{B}$-degrees b for which we gave equivalent characterizations in Theorem 4. The structure of R is determined by

Theorem 8. *Assume that $\mathfrak{B}$ is weakly inadmissible. Let S be the set of r.e. degrees in the admissible collapse $\mathfrak{A}$ of $\mathfrak{B}$. Then there exists an isomorphism $I: S \to R$, i.e. I maps S 1-1 onto R and for all $a, a' \in S$ we have $a \leqslant_{\mathfrak{A}} a' \leftrightarrow I(a) \leqslant_{\mathfrak{B}} I(a')$.*

We have that $0_{\mathfrak{B}} <_{\mathfrak{B}} I(0'_{\mathfrak{A}}) <_{\mathfrak{B}} 0'_{\mathfrak{B}}$ and for any $\mathfrak{B}$-degree b we have that $C \leqslant_{w\mathfrak{B}} b \leftrightarrow I(0'_{\mathfrak{A}}) \leqslant_{\mathfrak{B}} b$, where $C \in 0'_{\mathfrak{B}}$ is the universal Σ_1 set in $\mathfrak{B}$.

Further, there is a choice function F, which picks out of every degree b in R a t.r.e. regular set $F(b) \in b$ such that for all $a, a' \in S$

$$a \leqslant_{\mathfrak{A}} a' \leftrightarrow F(I(a)) \leqslant_{\mathfrak{B}} F(I(a')) \leftrightarrow F(I(a)) \leqslant_{w\mathfrak{B}} F(I(a')),$$

which implies that $F(I(a))$ is recursive iff $a \equiv 0_{\mathfrak{A}}$.

Proof. We pointed out before Theorem 4 that every $a \in S$ contains a β-immune $\mathfrak{A}$-r.e. set A. $I(a)$ is then defined as the degree of A in $\mathfrak{B}$. Observe that every β-immune set is trivially t.r.e. in $\mathfrak{B}$. The definition is independent of the choice of the β-immune r.e. set $A \in a$ and I is 1-1 because $A =_{\mathfrak{A}} A' \leftrightarrow A =_{\mathfrak{B}} A'$ for β-immune sets A, A'. Theorem 4 tells us that I maps S onto R.

It is easy to see that $C \leqslant_{w\mathfrak{B}} I(0'_{\mathfrak{A}})$, therefore $I(0'_{\mathfrak{A}}) \leqslant_{\mathfrak{B}} b \to C \leqslant_{w\mathfrak{B}} b$ for any $\mathfrak{B}$-degree b. For the other direction take a recursive set $A \in I(0'_{\mathfrak{A}})$. Define

$$A' := \{\langle 0, K\rangle \mid K \cap A \neq \emptyset\} \cup \{\langle 1, K\rangle \mid K \cap S_\beta - A \neq \emptyset\}.$$

We have then that for any $K \in S_\beta$:

$$K \subseteq A \leftrightarrow \langle 1, K\rangle \in S_\beta - A' \leftrightarrow \langle e, \langle 1, K\rangle\rangle \in S_\beta - C$$

and

$$K \subseteq S_\beta - A \leftrightarrow \langle 0, K\rangle \in S_\beta - A' \leftrightarrow \langle e, \langle 0, K\rangle\rangle \in S_\beta - C,$$

where e is some fixed index.

Friedman ([2], Theorem 2.4) showed that C is not recursive in any recursive $\mathfrak{B}$-degree (because the complete Σ_2 set is weakly recursive in C and any set which is weakly recursive in a recursive set is $\Delta_2\mathfrak{B}$), therefore we have $\neg C \leqslant_{\mathfrak{B}} I(0'_{\mathfrak{A}})$.

The choice function F is given by Theorem 4.

We can now directly apply results from admissible recursion theory in order to get information about recursive degrees in $\mathfrak{B}$. For example the following Corollary follows from Shore's splitting theorem [9] and the preceding:

Corollary. *Let $\mathfrak{B}$ be weakly inadmissible. Then the following holds.*

(a) *Take any recursive degree c in $\mathfrak{B}$ and any nonrecursive r.e. set $D \subseteq S_\beta$. Then* *there exist recursive* $\mathfrak{B}$*-degrees* a, b *such that* $c = \text{lub}\{a, b\}$, $\neg D \leqslant_{w\mathfrak{B}} a$ *and* $\neg D \leqslant_{w\mathfrak{B}} b$.

(b) (*Solution of Post's problem for recursive degrees with respect to* $\leqslant_{\mathfrak{B}}$.) *Take any recursive* $\mathfrak{B}$*-degree* $d \neq 0_{\mathfrak{B}}$. *Then there exist recursive* $\mathfrak{B}$*-degrees* a, b *such that* $d = \text{lub}\{a, b\}$, $\neg a \leqslant_{\mathfrak{B}} b$ *and* $\neg b \leqslant_{\mathfrak{B}} a$.

(c) (*Solution of Post's problem with respect to* $\leqslant_{w\mathfrak{B}}$.) *Take any recursive* $\mathfrak{B}$*-degree* $d \neq 0_{\mathfrak{B}}$. *Then there exist t.r.e. regular sets* A_1, A_2 *such that* $\neg A_1 \leqslant_{w\mathfrak{B}} A_2$, $\neg A_2 \leqslant_{w\mathfrak{B}} A_1$ *and* d *is the least upper bound of the degrees of* A_1 *and* A_2.

Remark 9. Friedman [2] has already proved (by performing a priority construction in S_β similar to Shore's blocking proof [11]): in weakly inadmissible $\mathfrak{B}$ there exist t.r.e. regular sets A_1, A_2 such that $\neg A_1 \leqslant_{w\mathfrak{B}} A_2$ and $\neg A_2 \leqslant_{w\mathfrak{B}} A_1$. Friedman has further observed in [2] that for any inadmissible β we can find a β-recursive set A such that every t.r.e. and every β-recursive set is recursive in A and such that $0 <_\beta A <_\beta 0'_\beta$.

Remark 10. It is described in [5] how the admissible collapse can be used to extend Shore's result about minimal degrees [10] to more admissible α and to show that minimal degrees and minimal pairs of degrees exist in weakly inadmissible $\mathfrak{B}$ for which the corresponding construction succeeds in the admissible collapse of $\mathfrak{B}$. Observe that it follows from the preceding and Theorem 7 in [5], that we have in fact degrees $a, b \in R$ such that $a \neq 0_{\mathfrak{B}}$ and $b \neq 0_{\mathfrak{B}}$ but $\text{glb}\{a, b\} \equiv 0_{\mathfrak{B}}$ if the admissible collapse is not refractory. Theorem 8 in [5] shows in fact that a $\mathfrak{B}$-degree b exists such that $b \leqslant_{\mathfrak{B}} I(0'_{\mathfrak{A}})$, $b \neq 0_{\mathfrak{B}}$ and such that for any $\mathfrak{B}$-degree d we have that $d \leqslant_{\mathfrak{B}} b \rightarrow d \equiv 0_{\mathfrak{B}} \vee d \equiv b$ if $\Sigma_2\text{-cf}\,\mathfrak{B} \geqslant \Sigma_2\text{-p}\,\mathfrak{B}$. This minimal degree b is an example of a degree which is recursive in a recursive degree but which is not recursive (by the preceding Corollary).

The relations between the considered stronger notions of $\mathfrak{B}$-r.e. degrees are now clear for weakly inadmissible $\mathfrak{B}$ (the t.r.e., s.r.e., n-r.e. ($n \geqslant 1$), n'-r.e. ($n \geqslant 1$) β-degrees are just the β-recursive degrees), but we don't know yet the exact relations between these notions for a single $\mathfrak{B}$-r.e. set. It is obvious that for any $A \subseteq S_\beta$: A 1-r.e. $\rightarrow A$ t.r.e., in fact we have A 1′-r.e. $\rightarrow A$ t.r.e. Friedman [2] asked whether A t.r.e. $\rightarrow A$ 1-r.e. We show after Theorem 11 that this is not the case. Nevertheless one can find a place for the (more recursion theoretic) notion t.r.e. in the (more syntactical) notion hierarchy n'-r.e., because we have A t.r.e. $\leftrightarrow A$ 1′-r.e. in any $\mathfrak{B}$.

We consider formulae of the form $\forall x \in p\, M(x, A, \boldsymbol{p})$ where $p, \boldsymbol{p}$ are parameters from S_β and m does not contain any quantifier and contains the predicate A only positively. $M(x, A, \boldsymbol{p})$ is then equivalent to a conjunction $M_1(x, A, A, \boldsymbol{p}) \wedge \cdots \wedge M_n(x, A, \boldsymbol{p})$ where every $M_i(x, A, \boldsymbol{p})$ has the form $\Psi(x, \boldsymbol{p}) \vee \boldsymbol{p} \in A \vee x \in A$ where $\Psi(x, \boldsymbol{p})$ is a quantifier free and does not contain A. The formulae of the form $\forall x \in p\, M_i(x, A, \boldsymbol{p})$ which are true in $\langle \mathfrak{B}, A \rangle$ can then be enumerated because we can split p into two pieces $p', p'' \in S_\beta$ such that

$$\langle \mathfrak{B}, A\rangle \models [\forall x \in p'\ \Psi(x, \boldsymbol{p}) \wedge \forall x \in p''(x \in A)]$$

(if $P_j \in A$ is not already true for some parameter $p_j \in \boldsymbol{p}$). This shows A t.r.e. $\rightarrow A$ 1′-r.e. (there is no problem with existential formulae $\exists x \in p\ M(x, A, \boldsymbol{p})$).

Theorem 11. *Let $\mathfrak{B}$ be weakly inadmissible and $A \subseteq \beta$ be r.e. Then there exist t.r.e. sets M_0, M_1 such that $M_0 \cap M_1 = \emptyset$ and $M_0 \cup M_1 = A$.*

Proof. We split A by a simple priority argument in such a way, that for every $K \in S_\beta$, $K \subseteq M_0$ or $K \subseteq M_1$ implies that K has cardinality (in $\mathfrak{B}$) less than κ. Then M_0 and M_1 are automatically t.r.e. if they are r.e.

let $g: \kappa \rightarrow A$ be an 1-1 onto recursive function (if such a g with dom $g = \kappa$ doesn't exist we have $A \in S_\beta$ and define $M_0 := A$). We take a recursive projection p of $\beta \times \{0, 1\}$ into $\mathfrak{B}^*$ and a recursive function $\eta \rightarrow K_\eta$ from β onto S_β. $q: \kappa \rightarrow \beta$ is a cofinal function as before.

At step $\delta < \kappa$ of the construction we choose $\eta \in \beta$ and $i \in \{0, 1\}$ such that
(a) $g(\delta) \in K_\eta$ and
(b) all elements of $g[\delta] \cap K_\eta$ have been put into M_i and
(c) if $\sigma_0 =$ the least $\sigma < \kappa$ where $\sigma \geq \delta$ and

$$\langle S_{q(\sigma)}, S_{q(\sigma)} \cap B\rangle \models (p(\langle \eta, i\rangle)\downarrow),$$

then no $\eta' \in \beta$, $i' \in \{0, 1\}$ exist such that

$$\langle S_{q(\sigma_0)}, S_{q(\sigma_0)} \cap B\rangle \models (p(\langle \eta', i'\rangle)\downarrow), \quad p(\langle \eta', i'\rangle) < p(\langle \eta, i\rangle),\ g(\delta) \in K_{\eta'},$$

and all elements of $g[\delta] \cap K_{\eta'}$ have been put into $M_{i'}$.
We say then that $\langle \eta, i\rangle$ receives attention at step δ. We put $g(\delta)$ into M_{1-i} and proceed to the next step.

In order to show that this construction succeeds we assume for a contradiction that for some $\eta \in \beta$, $i \in \{0, 1\}$ we have that $K_\eta \subseteq M_i$ and K_η has cardinality $\geq \kappa$ in $\mathfrak{B}$. Choose $\langle \eta, i\rangle$ with this property such that $p(\langle \eta, i\rangle)$ is minimal. Choose $\delta_0 < \kappa$ such that

$$\langle S_{q(\delta_0)}, S_{q(\delta_0)} \cap B\rangle \models (p(\langle \eta, i\rangle)\downarrow).$$

Then there exists an unbounded r.e. set of steps $\delta' > \delta_0$ such that $g(\delta') \in K_\eta$, $g[\delta'] \cap K_\eta \subseteq M_i$ and some $\langle \eta', 1-i'\rangle$ with $p(\langle \eta', 1-i'\rangle) < p(\langle \eta, i\rangle)$ receives attention at δ'. This yields a recursive projection of κ into $p(\langle \eta, i\rangle)$, contradicting $p(\langle \eta, i\rangle) < \mathfrak{B}^*$.

Corollary 1. *The union of two t.r.e. sets is in general not t.r.e.*

Corollary 2. *Assume $\mathfrak{B}$ is weakly inadmissible. Then there exist recursive sets $A \subseteq \beta$ which are not t.r.e. and there exist recursive t.r.e. sets $D \subseteq \beta$ such that D is neither 1-r.e. nor 2'-r.e.*

Proof. Take a $\mathfrak{B}$-recursive set $A' \subseteq \beta$ which is not an element of $0_{\mathfrak{B}}$. Then either A' or $\beta - A'$ is not t.r.e.

In order to get the set D we use Theorem 11 to split A into two t.r.e. sets M_0, M_1 such that $A = M_0 \cup M_1$, $M_0 \cap M_1 = \emptyset$. Then we define $D := M_0 \times \{0\} \cup M_1 \times \{1\}$. We are using here the pairing function for ordinals inside S_κ, which is of course an element of S_β. D is obviously t.r.e. and the idea is to show that D 1-r.e. or D 2'-r.e. would imply that A is t.r.e. This is easy to see, because we have for any $K \in S_\beta$:

$$K \subseteq A \leftrightarrow K \subseteq \kappa \wedge \forall x \in \{\{\langle\gamma, 0\rangle, \langle\gamma, 1\rangle\} \mid \gamma \in K\} \forall y \in \{\langle\gamma, 0\rangle \mid \gamma \in K\} \forall z \in \{\langle\gamma, 1\rangle \mid \gamma \in K\} ((y \in x \wedge z \in x) \rightarrow (y \in D \vee z \in D))$$

and

$$K \subseteq A \leftrightarrow K \subseteq \kappa \wedge \forall x \in \{\{\langle\gamma, 0\rangle, \langle\gamma, 1\rangle\} \mid \gamma \in K\} \exists y \in x\, (y \in D).$$

Remark 12. Friedman [2] observed that the Sacks–Simpson Lemma (Lemma 2.3 in [13]), which is an important tool for priority constructions, might fail for $\kappa = \Sigma_1\text{-cf}\,\mathfrak{B}$ in weakly inadmissible $\mathfrak{B}$, even if κ is a regular cardinal in $\mathfrak{B}$. If one takes the admissible collapse $\mathfrak{A}$ of $\mathfrak{B}$ in order to analyze this situation, one sees immediately that this failure occurs iff $\Sigma_2\text{-cf}\,\mathfrak{B} < \kappa$.

We have $\Sigma_2\text{-cf}\,\mathfrak{B} = \Sigma_2\text{-cf}\,\mathfrak{A}$. The failure defines a function which is Σ_2 over $\mathfrak{A}$ and cofinal in κ. If $\Sigma_2\text{-cf}\,\mathfrak{A} < \kappa$ one constructs a failure by applying Lemma 3.1 of Lerman–Simpson [4].

The failure usually doesn't bother us if we choose priorities in $\text{tp}\,2\,\mathfrak{B}$ (the tame Σ_2-projectum of $\mathfrak{B}$) for the priority construction. The reason is that we have $\text{tp}\,2\,\mathfrak{B} \leq \rho \cdot \Sigma_2\text{-cf}\,\mathfrak{B}$ in the relevant case where a greatest β-cardinal $\rho < \kappa$ exists. This follows because we have $\text{tp}\,2\,\mathfrak{A} \leq \text{gc}\,\mathfrak{A} \cdot \Sigma_2\text{-cf}\,\mathfrak{A}$ in the admissible $\mathfrak{A}$ ([14]), $\text{tp}\,2\,\mathfrak{B} \leq \text{tp}\,2\,\mathfrak{A}$ and $\rho = \text{gc}\,\mathfrak{A}$.

Remark 13. If $M \subseteq \kappa$ is a maximal set in the admissible collapse $\mathfrak{A}$, then $P^{-1}[M]$ is a maximal set in $\mathfrak{B}$.

Remark 14. The results of this paragraph can be generalized to a larger class of weakly inadmissible structures.

If $B \subseteq \text{L}_\beta$ is not regular over L_β we consider the structure $\mathfrak{B} = \langle \text{L}_\beta[B], \in, B\rangle$ where $\langle \text{L}_\gamma[B] \mid \gamma \in \text{On}\rangle$ is the constructible hierarchy relativized to B (see [1]). Then $\kappa := \Sigma_1\text{-cf}\,\mathfrak{B}$ and $\mathfrak{B}^*$ are still well defined and we assume that $\kappa \geq \mathfrak{B}^*$. In this more general situation we can't prove that $K \in L_\beta[B] \wedge K \subseteq \sigma$ for some $\sigma < \kappa \rightarrow K \in L_\kappa[B]$. Therefore the predicate T, which is defined as before, need not be regular over $L_\kappa[B]$. Therefore the admissible collapse is defined to be the structure $\mathfrak{A} := \langle L_\kappa[T], T\rangle$. $\mathfrak{A}$ is again admissible and we can proceed as before because we have $K \in L_\beta[B] \wedge K \subseteq \sigma$ for some $\sigma < \kappa \to K \in L_\kappa[T]$.

Example 15. If $\omega < \beta < \omega_1^{\mathrm{CK}}$ (ω_1^{CK} := the first nonrecursive ordinal), then β is weakly inadmissible with Σ_1-cf $\beta = \beta^* = \omega$.

Example 16. If α is admissible and $A \subseteq \alpha$ is α-r.e., regular and not complete, then the structure $\mathfrak{B} := \langle \mathrm{L}_\alpha, A\rangle$ is not strongly inadmissible ([15]). This example is relevant because we have that $B \subseteq \alpha$ is $\mathfrak{B}$-recursive iff $B \leqslant_{w\alpha} A$ and we have for sets $B, C \subseteq \alpha$ that $B \leqslant_{\mathfrak{B}} C$ iff $B \leqslant_\alpha A + C$ $(A + C := A \times \{0\} \cup C \times \{1\})$. $\mathfrak{B}$ is inadmissible iff A is not hyperregular.

Example 17. If α is admissible and $\alpha > \Sigma_2$-cf $\alpha \geqslant \Sigma_2$-p α, then $\langle \mathrm{L}_\alpha, C\rangle$ is weakly inadmissible where C is a complete regular Σ_1 set. A set $B \subseteq \alpha$ is r.e. in $\langle \mathrm{L}_\alpha, C\rangle$ iff B is Σ_2 definable over L_α.

Example 18. If β has the form $\rho + \rho$ where ρ is a regular cardinal in L, then β is weakly inadmissible.

Example 19. The following construction shows that gc $\beta > \max(\Sigma_1$-cf $\beta, \beta^*)$ can happen (gc β is the greatest β-cardinal).

Define $\alpha_0 = \omega_1^{\mathrm{L}}$ and

$$\alpha_{n+1} := \mu\gamma(\gamma > \alpha_n \wedge \mathrm{L}_\gamma <_{\Sigma_{n+1}} \mathrm{L}_{\omega_2 \mathrm{L}})$$

for all $n \in \omega$ and take $\beta := (\lim_{n\in\omega} \alpha_n) + \omega$. We have then Σ_1-cf $\beta = \omega$, $\beta^* = \omega_1^{\mathrm{L}}$ and gc $\beta = (\beta^*)^+ = (\lim_{n\in\omega} \alpha_n)$ ($(\beta^*)^+$ is the next β-cardinal after β^*).

Example 20. We get a weakly inadmissible set S_β with gc $\beta > \Sigma_1$-cf $\beta = \beta^* = \omega$ if we define $\alpha_0 = \omega$ and

$$\alpha_{n+1} := \mu\gamma(\gamma > \alpha_n \wedge \mathrm{L}_\gamma <_{\Sigma_{n+1}} \mathrm{L}_{\omega_1 \mathrm{L}})$$

and take again $\beta := (\lim_{n\in\omega} \alpha_n) + \omega$.

Example 21. If γ is not a gap ordinal, then $\beta := \omega \cdot \gamma + \omega$ is weakly inadmissible with Σ_1-cf $\beta = \beta^* = \omega$ (see Section 0).

2. Tame r.e. sets in strongly inadmissible β

So far not much is known about t.r.e. sets in strongly inadmissible β, in particular one doesn't know whether t.r.e. sets of nonzero degree exist. This question is of interest because Friedman [2] showed that for some strongly inadmissible β every t.r.e. set is of degree 0.

Neighborhood conditions like "$K \subseteq A$" are usually not considered in the fine structure theory of L and thus some notions of this theory are not very well suited for the study of t.r.e. sets. Especially Σ_1 projections are not "tame" enough to preserve the property "t.r.e." (i.e. if $f: \beta^* \to \beta$ is a Σ_1 projection and A is t.r.e. then $f^{-1}[A]$ is in general not t.r.e.). Therefore we introduce the notion of a tame projection and define the corresponding notion of a tame projectum of β (tp β). It turns out (Theorem 22) that this projectum has the same nice properties which the Σ_n-projectum of β has for every n according to Jensen's Uniformisation Theorem. It is plausible that the tame projectum of β was not considered so far because it is the same as β^* if β is admissible or weakly inadmissible. A further exploration of the tame projectum makes it possible to solve the problem concerning the existence of nonzero t.r.e. degrees (in Theorem 29). furthermore it is shown that s.r.e. sets of nonzero degree exist iff β is admissible or weakly inadmissible (see Theorem 26).

Let β be a limit ordinal and let $h: S_\delta \to S_\beta$ be a projection for some $\delta \leq \beta$ (i.e. h is 1-1 and onto but may be partial). We call h a tame projection if h (i.e. the graph of h) is t.r.e. and if $h \upharpoonright K$ is β-finite for every β-finite $K \subseteq \operatorname{dom} h$.

We write tp β for the tame projectum of β, which is the least $\delta \leq \beta$ such that a tame projection $h: S_\delta \to S_\beta$ exists. Analogously as for Σ_1-projections we further introduce

$\rho_\beta^t :=$ the largest $\delta \leq \beta$ such that all t.r.e. sets $B \subseteq S_\delta$ are regular over S_δ (we write always t.r.e. instead of β-t.r.e.),

$\pi_\beta^t :=$ the minimal $\delta \leq \beta$ such that a t.r.e. set $B \subseteq S_\delta$ exists which is not β-finite.

The letters κ and q have in this Section the same meaning as before.

Theorem 22. *Let β be any limit ordinal. Then we have*

a) *If $h: S_\delta \to S_\beta$ is a tame projection and $B \subseteq S_\beta$ is t.r.e. then $h^{-1}[B]$ is t.r.e. as well and we have $h^{-1}[B] \in S_\beta \to B \in S_\beta$ for any set $B \subseteq S_\beta$,*

b) *tp $\beta = \beta$ or tp β is a β-cardinal,*

c) *tp $\beta = \rho_\beta^t = \pi_\beta^t \geq \beta^*$,*

d) *tp $\beta = \beta^*$ if β is admissible or weakly inadmissible,*

e) *Σ_1-cf(tp $\beta)^{S_\beta} \leq \Sigma_1$-cf β.*

Proof.

a) We have $K \subseteq h^{-1}[B] \leftrightarrow \exists H \in S_\beta\ (H \subseteq h \wedge \operatorname{dom} H = K \wedge \operatorname{rg} H \subseteq B)$ for every β-finite set K.

b), c) Assume that $\pi_\beta^t < \beta$ and that π_β^t is not a β-cardinal. Then there exists a β-finite function g which maps $S_{\pi_\beta^t}$ 1-1 onto some S_γ where $\gamma < \pi_\beta^t$.
By definition of π_β^t there exists a t.r.e. set $B \subseteq S_{\pi_\beta^t}$ such that $\neg B \in S_\beta$.
The set $g[B] \subseteq S_\gamma$ has the same properties, contradicting the minimality of π_β^t.

We want to show that tp $\beta \leq \pi := \pi_\beta^t$. Assume that $\pi < \beta$. Since π is a β-cardinal, π is admissible. For the following definition of the function g we take an onto pairing function $\langle\,,\rangle : \pi \times \pi \to \pi$ which is Σ_1 definable over S_π. We write p_0, p_1 for the accompanying projections.

Take a t.r.e. set $B \subseteq \pi$ such that $\neg B \in S_\beta$. Then there exists a partial recursive function $f: S_\beta \to B$ which is 1-1 onto such that

$$\forall K \in S_\beta\,(K \subseteq B \to \exists \sigma\,(K \subseteq f[S_\sigma]))$$

and such that the function $\sigma \to f \cap S_\sigma$ is recursive and maps β into S_β (see [2] for similar enumeration functions). $\neg B \in S_\beta$ implies that dom f is unbounded in $<_{S_\beta}$ ($<_{S_\beta}$ is the natural well ordering of S_β, $<_{S_\beta}$ is uniformly Δ_1 definable over S_β, see [1]). Let P be a recursive projection of S_β into β^*. We have $\beta^* \leq \pi$ because $B \subseteq \pi$ is r.e. and not β-finite. Define $g: S_\beta \to S_\pi$ by $g(z) = \langle f(z'), P(z)\rangle$ where $z' \geq_{S_\beta} z$ is $<_{S_\beta}$-minimal such that $z' \in \operatorname{dom} f$ and such that we have for the $\sigma \in \beta$ with $z' \in S_{\sigma+1} - S_\sigma$ that $S_{\sigma+1} \models \exists y\,(P(z) = y)$. Then dom $g = S_\beta$ and g is 1-1. We want to show that $h := g^{-1}$ is a tame projection. Take $K \in S_\beta$ such that $K \subseteq \operatorname{dom} h$. $K' := p_0[K] \in S_\beta \to \exists \sigma\,(K' \subseteq f[S_\sigma]) \to$ the function H with $H \subseteq h$, dom $H = K$ is β-finite. h, considered as a set of pairs, is a t.r.e. set because we have for $K \subseteq h$ that $\exists \sigma\,(p_0[\operatorname{dom} K] \subseteq f[S_\sigma])$. This shows that tp $\beta = \pi^t_\beta$ (tp $\beta \geq \pi^t_\beta$ follows from a) with $B := S_\beta$).

$\rho^t_\beta \leq \pi^t_\beta$ is obvious. In order to show $\rho^t_\beta = \pi^t_\beta$, we consider a t.r.e. set $B \subseteq \pi^t_\beta$. For $\sigma \in \kappa$ we have $B \cap \sigma \in S_\beta$, due to the definition of π^t_β. This implies $B \cap \sigma \in S_{\pi_\beta{}^t}$ because π^t_β is a β-cardinal.

d) The claim is trivial if β is admissible or if $\beta^* < \Sigma_1\text{-cf}\,\beta < \beta$ (consider π^t_β). If $\beta^* = \Sigma_1\text{-cf}\,\beta < \beta$ we take any β-immune r.e. set $B \subseteq \beta^*$ such that B is not recursive and get $\beta^* = \pi^t_\beta$.

e) Take a tame projection $h: \text{tp}\,\beta \to S_\beta$ and a cofinal recursive function $q: \Sigma_1\text{-cf}\,\beta \to \beta$. Consider $k := h^{-1} \cdot q$ and assume for a contradiction that $\exists \sigma <$ tp β (rg $k \subseteq \sigma$). Since $h[\sigma]$ is bounded this would imply that rg q is bounded.

Remark 23. The definition of a tame projection has some similarity to the definition of a tame Σ_2 projection in admissible recursion theory (see e.g. [14]). Unfortunately the two notions do not fit together. It is essential that the tame projectum tp β is a β-cardinal if tp $\beta < \beta$. On the other hand it is important for applications of the tame Σ_2 projection in priority constructions that we have in some cases tp 2 $\alpha < \alpha$ even if we can't project any α-cardinal "tamely" onto α (i.e. gc $\alpha <$ tp 2 α).

Lemma 24. *Assume β is strongly inadmissible and $B \subseteq S_\beta$ is t.r.e. and regular. Then we have $B \leq_\beta \emptyset$.*

Proof. Take a (partial) function $f: S_\beta \to B$ which enumerates B as in the preceding proof. We define $M \subseteq \kappa \times \kappa$ ($\kappa := \Sigma_1\text{-cf}\,\beta$) by

$$\langle \eta, \delta\rangle \in M \leftrightarrow \forall x \in S_{q(\eta)}\,(x \in B \to x \in f[S_{q(\delta)}]),$$

where we use pairing in the admissible κ. M has obviously a Π_1 definition over S_β and since $\kappa - M$ is bounded below β^* we have that $M \in S_\beta$. The function $g: \kappa \to \kappa$ such that $g(\eta) = \mu\delta(\langle\eta, \delta\rangle \in M)$ is therefore β-finite as well. Then we have for any $K \in S_\beta$:

$$K \subseteq S_\beta - B \leftrightarrow \exists\eta\ (K \subseteq S_{q(\eta)} \wedge K \cap f[S_{q(g(\eta))}] = \emptyset).$$

which shows $B \leqslant_\beta \emptyset$.

Lemma 25. *Assume β is strongly inadmissible,* tp $\beta < \beta$ *and $B \subseteq S_{\mathrm{tp}\,\beta}$ is t.r.e. Then the set $\{K \in S_{\mathrm{tp}\,\beta} \mid K \subseteq S_{\mathrm{tp}\,\beta} - B\}$ is r.e.*

Proof. B is regular over $S_{\mathrm{tp}\,\beta}$ and there exists a cofinal recursive function $h: \kappa \to \mathrm{tp}\,\beta$ (by Theorem 22). Take an enumeration f of B as before and a β-finite function $g: \kappa \to \kappa$ analogously as in Lemma 24 such that for all $\delta \in \kappa$, $B \cap S_{h(\delta)} \subseteq f[S_{q(g(\delta))}]$. Proceed then as in Lemma 24.

Theorem 26. *Let β be strongly inadmissible. Then we have*

a) *If $A \subseteq S_\beta$ is t.r.e. then A is recursive,*
b) *If $A \subseteq S_\beta$ is 2'-r.e. (e.g. if A is s.r.e.) then $A \leqslant_\beta \emptyset$,*
c) $\Sigma_1\text{-cf}\ \beta = \Sigma_1\text{-cf}\ (\mathrm{tp}\ \beta)^{S_\beta}$.

Proof. a) By Lemma 24 we may assume that A is not regular over S_β which implies that $\mathrm{tp}\,\beta < \beta$. Take a tame projection $h: S_{\mathrm{tp}\beta} \to S_\beta$ and consider $B := h^{-1}[A]$. then B is t.r.e. and A is recursive since B is recursive by Lemma 25.

b) Consider $A' := \{K \in S_\beta \mid K \cap A \neq \emptyset\}$. Since A is 2'-r.e. we have that A' is t.r.e. and $A =_\beta A'$. Then A' is recursive and we have that $K \subseteq S_\beta - A \leftrightarrow \neg K \in A'$.

c) Assume that $\mathrm{tp}\,\beta < \beta$ and take a tame projection $h: \mathrm{tp}\,\beta \to S_\beta$. Let $g: \Sigma_1\text{-cf}\ (\mathrm{tp}\ \beta) \to \mathrm{tp}\ \beta$ be a cofinal recursive function. Then the function $p: \Sigma_1\text{-cf}\ (\mathrm{tp}\ \beta) \to \beta$ which is defined by $p(\delta) = \mu\sigma(h[g(\delta)] \subseteq S_\sigma)$ is cofinal in β and recursive because of Lemma 25 (observe that h is always partial).

Lemma 27. *Assume that β is strongly inadmissible and let $\rho \geqslant \beta^*$ be a β-cardinal such that $\Sigma_1\text{-cf}\ \beta = \Sigma_1\text{-cf}\ \rho$. Then there exists a recursive function $g: \kappa \to S_\rho$ such that*

$$\forall K \in S_\beta\ (K \subseteq g \to \exists\sigma < \kappa\ (\mathrm{dom}\ K \subseteq \sigma))$$

(which implies that the graph of g is t.r.e.) and such that the complete Σ_1 set C is weakly recursive in g (which implies that $\neg g \leqslant_\beta \emptyset$).

Proof. Take a Σ_1 formula Φ which defines C over S_β. Let $p: \kappa \to \rho$ be a recursive strictly increasing cofinal function and let $f: \beta \to \beta^*$ be a recursive projection. Define g by

$$g(\delta) := \{x \in S_{p(\delta)} \mid S_{q(\delta)} \models \exists y\ (\Phi(y) \wedge f(y) = x)\},$$

where $q:\kappa \to \beta$ is cofinal and recursive. Take any $K \in S_\beta$ such that $K \subseteq g$ and assume for a contradiction that dom K is unbounded in κ. Then $H := \bigcup(\text{rg } K)$ is β-finite and we have $\neg\, y \in C \leftrightarrow f(y) \notin H$ which yields the contradiction that C is recursive. Finally we have $C \leqslant_{w\beta} g$ because

$$y \notin C \leftrightarrow \exists x\,(f(y) = x \wedge \exists H \in S_\beta\,(H = \{K \in S_\rho \mid x \in K\} \wedge \kappa \times H \subseteq S_\beta - g).$$

Lemma 28. *Assume that β is strongly inadmissible and* gc $\beta > \beta^*$. *Then we have* tp $\beta \leqslant (\beta^*)^+$ (($\beta^*)^+ :=$ *the next β-cardinal after β^**).

Proof. Take C, Φ, f as in Lemma 27 and define $g:\kappa \to S_{(\beta^*)^+}$ by

$$g(\delta) := \{x \in \beta^* \mid S_{q(\delta)} \models \exists y\,(\Phi(y) \wedge f(y) = x)\}.$$

It follows as in Lemma 27, that

$$\forall K \in S_\beta\,(K \subseteq g \to \exists \sigma < \kappa\,(\text{dom } K \subseteq \sigma),$$

which implies that $\neg\, g \in S_\beta$ and that the graph of g is t.r.e. Since $g \subseteq S_{(\beta^*)^+}$ we get tp $\beta \leqslant (\beta^*)^+$ by Theorem 22.

Theorem 29. *Assume that β is strongly inadmissible. Then we have*

a)

$$\text{tp}\,\beta < \beta \leftrightarrow (\Sigma_1\text{-cf}\,\beta = \Sigma_1\text{-cf}\,(\beta^*)^{S_\beta} \wedge \text{gc}\,\beta > \beta^*),$$

where

$$\text{tp}\,\beta = \beta^* \quad \text{iff} \quad \Sigma_1\text{-cf}\,\beta = \Sigma_1\text{-cf}\,(\beta^*)^{S_\beta}$$

and

$$\text{tp}\,\beta = (\beta^*)^+ \quad \text{iff} \quad \Sigma_1\text{-cf}\,\beta \neq \Sigma_1\text{-cf}\,(\beta^*)^{S_\beta},$$

b) tp $\beta < \beta \leftrightarrow$ *there exists a t.r.e. set A such that $\neg A \leqslant_\beta \phi$,*

c) *if* tp $\beta < \beta$ *then* tp β *is the least ordinal δ such that a t.r.e. set $A \subseteq S_\delta$ exists which is of nonzero degree,*

d) *if* tp $\beta < \beta$, *then there exists a t.r.e. set $A \subseteq S_{\text{tp}\,\beta}$ such that every recursive set (especially every t.r.e. set) is recursive in A and such that the complete Σ_1 set C is weakly recursive in A.*

Proof. a) if tp $\beta = \beta^*$ then we have Σ_1-cf $(\beta^*)^{S_\beta} = \Sigma_1$-cf β by Theorem 26. If Σ_1-cf $(\beta^*)^{S_\beta} = \Sigma_1$-cf β then Lemma 27 ($\rho := \beta^*$) shows that tp $\beta = \beta^*$. Further, if Σ_1-cf $\beta \neq \Sigma_1$-cf $(\beta^*)^{S_\beta}$ and gc $\beta > \beta^*$, we have tp $\beta \neq \beta^*$ by Theorem 26 and therefore tp $\beta = (\beta^*)^+$ by Lemma 28 because tp β is a β-cardinal.

b), c), d) Assume tp $\beta < \beta$. A t.r.e. set A with all the properties which are required in d) is given by the graph of g in Lemma 27 (if tp $\beta = \beta^*$) and by the graph of g in Lemma 28 (if tp $\beta = (\beta^*)^+$). That every recursive set is recursive in g follows from $C \leqslant_{w\beta} g$ (see the proof of Theorem 8). Finally, if t.r.e. set A exists such that $\neg A \leqslant_\beta \emptyset$, then A can't be regular by Lemma 24. Therefore there exists a $\sigma < \beta$ such that $A \cap S_\sigma \notin S_\beta$, which shows that tp $\beta < \beta$ by Theorem 22.

Remark 30. $\beta := \omega_1^{\mathrm{L}} + \omega$ is an example for tp $\beta = \beta$. $\beta := \omega_\omega^{\mathrm{L}} + \omega$ is an example for tp $\beta = \beta^*$. Example 19 in Section 1 is an example for tp $\beta = (\beta^*)^+$. If we change Example 19 slightly (take $\alpha_0 := \omega_\omega^{\mathrm{L}}$, consider substructures of $\omega_{\omega+1}^{\mathrm{L}}$ instead of ω_2^{L}) we get an example for gc $\beta >$ tp $\beta = \beta^*$.

Remark 31. We have proved now, that for every limit ordinal β a t.r.e. β-degree t exists, which is an upper bound for all t.r.e. β-degrees, and that

(i) $t \equiv 0'_\beta$, if β is admissible,
(ii) $t \equiv$ the largest recursive β-degree, if β is inadmissible and tp $\beta < \beta$,
(iii) $t \equiv 0_\beta$, if β is inadmissible and tp $\beta = \beta$.

Remark 32. The three classes of limit ordinals (admissible, weakly inadmissible, strongly inadmissible) can be characterized by the solvability of versions of Post's problem. Consider the following three versions.

(P1) There exist regular t.r.e. sets A, B such that $\neg A \leqslant_\beta B$ and $\neg B \leqslant_\beta A$.
(P2) There exist t.r.e. sets A, B such that $\neg A \leqslant_{w\beta} B$ and $\neg B \leqslant_{w\beta} A$.
(P3) There exist t.r.e. degrees a, b such that for every $A \in a, B \in b$: $\neg A \leqslant_{w\beta} B$ and $\neg B \leqslant_{w\beta} A$.

It follows from Sacks–Simpson [12] and the preceding theorems that (P1) is solvable for β iff β is admissible or weakly inadmissible (the same holds for (P2)) and that (P3) is solvable for β iff β is admissible.

Remark 33. Friedman [2] showed that for example in the case where β is strongly inadmissible and β^* is a successor cardinal of L we have that every regular β-r.e. set is of degree 0. On the other hand the preceding theorems show that t.r.e. sets of nonzero degree may exist in such a β (e.g. take Example 19) in Section 1).

Remark 34. There are examples of strongly admissible β where regular r.e. sets of nonzero degree exist. Assume tp $\beta = \beta^*$ and take functions $g : \kappa \to \beta^*$, $q : \kappa \to \beta$ which are recursive, cofinal, strictly increasing and continuous ($\kappa := \Sigma_1$-$cf\,\beta$). Let C be the complete Σ_1 set and let $P : S_\beta \to \beta^*$ be a recursive projection. Define

$$\langle \sigma, x \rangle \in A \leftrightarrow \exists \delta \in \kappa \; (\sigma = q(\delta) \wedge g(\delta) \leqslant x < g(\delta+1) \wedge P^{-1}(x) \in C).$$

Then A is regular, r.e. and of nonzero degree because $C \leqslant_{w\beta} A$.

Remark 35. The last two remarks together with Theorem 26a), b) show that many conjectures concerning a regular set theorem for strongly inadmissible β are either trivial or false.

3. On the interpretation of some basic notions

In β-recursion theory the "finite" sets are the elements of S_β (respectively L_β). It is hardly possible to capture in one definition all properties of finite sets in ORT. For example if one wants to preserve the property in β-recursion theory that the recursive predicates are closed under quantification over finite sets (this property is preserved in α-recursion theory) or if one looks at the structure of the lattice of β-r.e. sets with inclusion this may lead to the consideration of alternative definitions of "finite" (e.g. elements of S_β with a "small" β-cardinality).

For the following we consider only the usual definition of "finite" because we want to discuss instead the definition of "r.e." and "recursive in". There is an alternative concerning the definition of a "r.e. set" A because one has to decide whether one wants that

(1) the elements $x \in A$ or that

(2) the positive neighborhood facts $K \subseteq A$ (with K β-finite)

are enumerable by an enumeration process.

Whereas in ORT and α-recursion theory (1) and (2) lead to the same definition one has in β-recursion theory the choice between β-r.e. (with (1)) and β-tame r.e. (with (2)). Friedman and Sacks have chosen alternative (1). With this choice one doesn't narrow the view in advance and one can still consider t.r.e sets as a special class of r.e. sets. In fact the relation between t.r.e and r.e. sets seems to be one of the most interesting new problems in β-recursion theory.

We want to stress here that after the choice of (1) the reducibility relation splits into two relations which have to be distinguished carefully. It makes sense to define "A is recursive in B" such that this holds iff A and the complement of A are "r.e. in B". Now the definition of "r.e. in B" should be consistent with the definition of "r.e." and should just bring in the additional feature that we may ask questions about B during the enumeration process. Here the further problem arises of what sort of questions we should allow about the oracle B during this enumeration process. A decision concerning this problem seems to be relatively independent from a decision between (1) and (2), because here we merely want to describe which "abilities" are required of an oracle. If we have in mind that every single computation is a β-finite object in β-recursion theory it seems very natural to allow exactly all questions $K \subseteq B$, $K \subseteq CB$ for β-finite sets K. Thus concerning the definition of "recursive in" we arrive in β-recursion theory at $\leqslant_{w\beta}$ with (1) (with (2) we would arrive at $\leqslant_\beta$ and no problem would occur). Of course the relation $\leqslant_\beta$ is more attractive because it is transitive. But we are not required to throw away this relation with (1) because $\leqslant_\beta$ is in any case the canonical definition for "A can always be replaced by B as an oracle". Therefore β-degrees are equivalence classes of oracles and it makes sense to study these equivalence classes.

Therefore two different reducibilities are of interest in β-recursion theory: $A \leqslant_{w\beta} B$ tells us that we can compute A if we assume that B is given. $A \leqslant_{\beta} B$ tells us that the oracle A can always be replaced by the oracle B.

Both relations are the same in ORT but the split occurred already in α-recursion theory (nevertheless it is often said that $\leqslant_{\alpha}$ is the definition of "recursive in" in α-recursion theory; an explanation might be that one has alternative (2) in mind). In β-recursion theory one stumbles immediately upon the fact that there exist β-recursive sets A such that $A \not\leqslant_{\beta} \emptyset$. Whereas it is absurd to say that the "recursive" set A is not "recursive in" the empty set it makes sense to say that the "recursive" set A is a stronger oracle than the empty set although this effect could not be demonstrated before β-recursion theory was started.

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References

[1] K.J. Devlin, Aspects of constructibility, Springer Lecture Notes, Vol. 354 (Springer, Berlin, 1973).
[2] S.D. Friedman, Recursion on inadmissible ordinals, Thesis, MIT, Cambridge, MA, 1976.
[3] L. Harrington, Contributions to recursion theory in higher types, Thesis, MIT, Cambridge, MA, 1973.
[4] M. Lerman and S.G. Simpson, Maximal sets in α-recursion theory, Israel J. Math. 4 (1973) 236–247.
[5] W. Maass, On minimal pairs and minimal degrees in higher recursion theory, Arch. Math. Logik 18 (1977) 169–186.
[6] W. Maass, The uniform regular set theorem in α-recursion theory, to appear in J. Symbolic Logic.
[7] W. Marek and M. Srebrny, Gaps in the constructible universe, Ann. Math. Logic 6, (1974) 359–394.
[8] G.E. Sacks, R.e. sets higher up, Proceedings of the Int. Congress for Logic, Philosophy and Methods of Science, 1975.
[9] R.A. Shore, Splitting an α-recursively enumerable set, Trans. Am. Math. Soc. 204 (1975) 65–78.
[10] R.A. Shore, Minimal α-degrees, Ann. Math. Logic 4 (1972) 393–414.
[11] R.A. Shore, Σ_n sets which are Δ_n incomparable (uniformly), J. Symbolic Logic 39 (1974) 295–304.
[12] S.G. Simpson, Admissible ordinals and recursion theory, Thesis, MIT, Cambridge, MA, 1971.
[13] G.E. Sacks and S.G. Simpson, The α-finite injury method, Ann. Math. Logic 4 (1972) 323–367.
[14] M. Lerman and G.E. Sacks, Some minimal pairs of α-recursively enumerable degrees, Ann. Math. Logic 4 (1972) 415–442.
[15] R.A. Shore, The recursively enumerable α-degrees are dense, Ann. Math. Logic 9 (1976) 123–155.