STRONG REDUCIBILITIES IN α- AND β-RECURSION THEORY

Martin Dietzfelbinger and Wolfgang Maass
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
Chicago, Illinois 60680

ABSTRACT

We start an investigation of strong reducibilities in α- and β-recursion theory. In particular, we study Myhill's Theorem about recursive isomorphisms ($A \leq_1 B$ & $B \leq_1 A \Leftrightarrow A \equiv B$), and show that it holds for a limit ordinal β if and only if $\sigma\mathrm{lcf}\beta = \omega$. (In particular, it fails for all admissible $\alpha > \omega$.) We point out a consequence for Σ^1_n-sets ($n \geq 2$) under $V = L$.

§1. INTRODUCTION AND BASIC DEFINITIONS

During the last twenty years classical recursion theory (CRT) has been extended to a theory of computable functions on admissible ordinals (α-recursion theory) respectively arbitrary limit ordinals (β-recursion theory).

These new theories concentrated on the study of Turing-degrees (e.g. Post's problem) and of the lattice of recursively enumerable sets. So far recursive isomorphisms and strong reducibilities $\leq_1$, $\leq_m$, etc. have not been considered in α- or β-recursion theory (except for a few elementary results on β-recursive isomorphisms in [14]). In this paper we begin a study of this latter subject.

A general experience has been, that results that can be proved rather easily in CRT (like e.g., the solution of Post's problem) can be generalized to all admissible ordinals and even to many inadmissible ordinals. On the other hand, results from CRT that require more complicated constructions (like e.g., minimal pairs) are more difficult to generalize, and might not even hold for all admissible ordinals.

We analyze in this paper an extremely easy (although important) result from classical recursion theory - Myhill's Theorem - and show that it holds for no admissible ordinal $\alpha > \omega$, further that it holds for arbitrary limit ordinals β if and only if $\sigma\mathrm{lcf}\beta = \omega$. By "Myhill's Theorem" we mean here the following result (see Rogers [16], Theorem 7-VI): Any sets A and $B \subseteq \omega$ are one-one reducible to each other (i.e., $A \leq_1 B$ and $B \leq_1 A$) if and only if A and B are recursively isomorphic ($A \equiv B$).

The notions $\leq_1$ and $\equiv$ which occur in Myhill's Theorem are well defined for arbitrary limit ordinals β. A subset of β (or of L_β)

is called β-recursively enumerable (β- r.e.) if and only if it is Σ_1-definable over L_β (L_β is the collection of all sets that appear in the hierarchy of constructible sets before level β, we refer the reader to Devlin [3] for details about constructible sets). A function f from β into β (f may be partial) is called β-recursive if and only if the graph of f is β- r.e. Thus for subsets A, B of β one says that $A \equiv B$ (A is β-recursively isomorphic to B) if and only if there is β-recursive function f that maps β one-one onto β such that $f(A) = B$. Further $A \leq_1 B$ (A is one-one reducible to B) if and only if there is a β-recursive one-one map f from β into β such that

$$\forall x \in \beta(x \in A \leftrightarrow f(x) \in B).$$

$A \equiv_1 B$ is an abbreviation for $A \leq_1 B$ and $B \leq_1 A$.

We would like to point out that for certain ordinals $\beta > \omega$ the concepts that are considered in the generalization of Myhill's Theorem to β coincide with well known notions from descriptive set theory, in case $V = L$. In particular for $\beta = \aleph_1^L$ (where $\aleph_1^L$ is the first uncountable L-cardinal) the real numbers in L can be identified with the ordinals less than β and a function is β-recursive if and only if the corresponding function from reals into reals is $\underset{\sim}{\Sigma}_2^1$-definable (further $\underset{\sim}{\Sigma}_n^1$-definable for $n > 2$ corresponds to Σ_1 over $\langle L_{\aleph_1^L}, S_n \rangle$ for some suitable mastercode S_n; our results on Myhill's Theorem remain valid for such admissible structures). Therefore the statement of Myhill's Theorem for $\beta = \aleph_1^L$ is equivalent to the question whether for all sets A, B of reals such that $A = f^{-1}[B]$ and $B = g^{-1}[A]$ for some one-one $\underset{\sim}{\Sigma}_2^1$ functions f and g there is a $\underset{\sim}{\Sigma}_2^1$ definable permutation h of the reals with $h[A] = B$. We give a negative answer to this question (under $V = L$). We can even show (via a priority argument) that there are $\underset{\sim}{\Sigma}_2^1$ sets A and B for which this statement does not hold.

This paper is largely self-contained. A reader that is only interested in α-recursion theory may substitute α for β throughout this paper. We use only very elementary notions from β-recursion theory, which we repeat here for completeness. One writes $\sigma\text{lcf}\beta$ for the least ordinal $\delta \leq \beta$ such that there is some β-recursive function whose domain is δ and whose range is unbounded in β (thus β is admissible if and only if $\sigma\text{lcf}\beta = \beta$). β^* is the least ordinal $\delta \leq \beta$ such that some β-recursive function maps β one-one into δ. $\hat{\beta}$ is the least ordinal $\delta \leq \beta$ such that some β-recursive function maps

β one-one onto δ (by Friedman [8] one has $\hat{\beta} = \max(\beta^*, \sigma lcf\beta)$ for all limit ordinals β). An ordinal $\delta < \beta$ is called a β-cardinal if no function $f \in L_\beta$ maps δ one-one into some $\gamma < \delta$. A set $\alpha \in L_\beta$ is called i-finite if and only if some function $f \in L_\beta$ maps α one-one onto some $\delta < \sigma lcf\beta$ (see [14] for other equivalent definitions).

For partial functions f and g we write $f(x) \simeq g(x)$ if and only if either f and g are defined on x and have the same value or both functions are not defined on x.

In section 2 of this paper we show that Myhill's Theorem fails for all β with $\sigma lcf\beta > \omega$ (in particular for all admissible $\alpha > \omega$). In section 3 we show that Myhill's Theorem holds if $\sigma lcf\beta = \omega$.

In section 4 we sketch the outline for a systematic development of the theory of strong reducibilities in α- and β-recursion theory. We show that Myhill's Theorem can be saved for all limit ordinals β if one considers the reducibility $\leq_1^r$ (where one demands that in addition the range of the reducing function is β-recursive) instead of $\leq_1$. We introduce an appropriate generalization of the notion of "acceptable Goedel numbering." We show for example that for all β a β-r.e. set is creative if and only if it is m-complete. Further if $\sigma lcf\beta \geq \beta^*$ these notions coincide with 1-completeness. More detailed proofs for results in section 4 can be found in the Diplomarbeit [4] of the first author.

§2. MYHILL'S THEOREM FAILS IF $\sigma lcf\beta > \omega$

Let two β-recursive functions $f,g\colon \beta \xrightarrow{1-1} \beta$ and two sets $A,B \subseteq \beta$ be given so that $A \equiv_1 B$ via f,g, i.e., $f^{-1}[A] = B$ and $g^{-1}[B] = A$. How can we find a β-recursive permutation $h\colon\beta \xrightarrow[\text{onto}]{1-1} \beta$ such that $A \equiv B$ via h? It does not make sense to define h in terms of A and B, since h is to be β-recursive, and nothing is said about the definability of A and B. So, given $x \in \beta$, which elements of β can we use as $h(x)$? We observe

$$x \in A \iff f(x) \in B \iff (fg)f(x) \in B \iff$$
$$\iff (fg)^k f(x) \in B \text{ for any (or all) } k \in \omega \iff$$
$$\iff \exists y \in B \exists k \in \omega\, ((fg)^k(y) = f(x)),$$

by the definition of $\equiv_1$.

We are thus led to considering the sets of all elements of β which can be reached from $f(x)$ by iterating fg or $(fg)^{-1}$. Analogously, to find some $h^{-1}(y)$ for some $y \in \beta$, we can choose from all $x \in \beta$ reachable from $g(y)$ by iterating gf or $(gf)^{-1}$.

2.1. Definition.

Let $x,x',y,y' \in \beta$.

$x \sim^A x'$ ("x and x' are in the same A-class"): $\Longleftrightarrow$
$(\exists k \in \omega)(x = (gf)^k(x') \vee x' = (gf)^k(x))$

$x \sim^B y'$ ("y and y' are in the same B-class"): $\Longleftrightarrow$
$(\exists k \in \omega)(y = (fg)^k(y') \vee y' = (fg)^k(y))$

$[x]^A := \{x' \in \beta | x' \sim^A x\}$

$[y]^B := \{y' \in \beta | y' \sim^B y\}$

A pair $([x]^A,[y]^B)$ is called an orbit if $f(x) \in [y]^B$ (iff $g(y) \in [x]^A$ iff $\exists k \in \omega(x = (gf)^k g(y) \vee y = (fg)^k f(x))$

Note. All these notions should carry a subscript "f,g", which we omit. The superscripts "A" and "B" do not indicate that the orbits depend on the sets A,B but only that they should be thought of as subsets of β, as the domain of h and f (A-classes) or as the range of h and domain of g (B-classes) respectively.

We state a trivial fact:

(*) $([x]^A \subseteq A \mathbin{\&} [f(x)]^A \subseteq B) \vee ([x]^A \cap A = [f(x)]^B \cap B = \emptyset)$, for all $x \in \beta$.

(In fact, (*) is equivalent to the definition of "$A \equiv_1 B$ via f,g").
Now the ideas discussed above can be made more precise as follows:

the members of $[f(x)]^B$ can serves as $h(x)$
the members of $[g(y)]^A$ can serve as $h^{-1}(y)$.

The familiar proof of Myhill's Theorem in CRT, as it can be found, e.g. in Rogers [16], works along these lines: Let $A,B \subseteq \omega$, $A \equiv_1 B$ via f,g. h is defined in ω stages:

Stage 2n. If n is already in dom(h), go to Stage 2n+1. Otherwise, look for some member of $[f(n)]^B$ not yet in ran(h) by inspecting $f(n)$, $(fg)f(n)$, etc. If y is the first suitable element encountered in that way, define $h(x) = y$.

Stage 2n+1. If n is already in ran(h), go to Stage 2n+2. Otherwise, choose analogously some $x \in [g(n)]^A$ not yet in dom(h) and define $h(x) = n$.

We observe one fact, which seems trivial, but is essential for the construction to work:

At every stage in the construction, if $([x]^A,[y]^B)$ is an infinite orbit, there are infinitely many elements of $[x]^A$ (resp. $[y]^B$), which have not yet entered dom(h) (resp. ran(h)). So you can be sure that at every stage of the construction you will be able to find a suitable counterpart for n.

Now, what happens if one tries to construct such an h in the same way for some admissible $\alpha > \omega$? Again, let $A,B \subseteq \alpha$, $A \equiv_1 B$ via f,g. One tries to construct h in α many steps. It is easily seen that $[x]^A$ and $[y]^B$ are α-finite sets (of α-cardinality ω or less), for all $x,y \in \alpha$. Since h is α-recursive and total, there is some stage at which h must be defined on all of $[x]^A$ (by admissibility). But now: how can you make sure that at this stage <u>all</u> elements of $[f(x)]^B$ are in ran(h)? Let us look at a special orbit to make this difficulty apparent: Assume there is some x_0 (we cannot compute it from x) such that $[x]^A = \{x_0,(gf)(x_0),\ldots\}$. Now there may be an element y_0 such that $g(y_0) = x_0$ or not. (y_0 would be another element of $[f(x)]^B$.) But at some stage you have to finish defining h on $[x]^A$. If some y_0 as described emerges after that stage, we have to choose $h^{-1}(y_0)$ outside of $[x]^A$, and it is no longer guaranteed that $y_0 \in B \Leftrightarrow h^{-1}(y_0) \in A$.

This feature of the enumerations of f and g, and the α-recursive isomorphisms - h has to settle down on every α-finite set of α-cardinality $\leq \alpha$, but orbits may change "later" - is used in the following to construct a counterexample to Myhill's Theorem for all $\alpha > \omega$.

<u>Remark</u>. In the case $\alpha^* < \alpha$ there exists a counterexample for trivial reasons: Split α into two α-recursive unbounded sets A and $\alpha - A$. Choose α-recursive mappings $f_1 : A \xrightarrow{1-1} \alpha^*$ and $f_2 : \alpha - A \xrightarrow{1-1} \alpha - \alpha^*$. ($f_1$ exists by the definition of α^*.) Choose any α-recursive functions $g_1 : \alpha^* \xrightarrow{1-1} A$ and $g_2 : \alpha - \alpha^* \xrightarrow{1-1} \alpha - A$. Then $A \equiv_1 \alpha^*$ via $f_1 \cup f_2$, $g_1 \cup g_2$, but $A \equiv \alpha^*$ is impossible, since α^* is α-finite and A is onto.

The following theorem provides counterexamples for all admissibles $\alpha > \omega$. The counterexamples produced there for the case $\alpha^* < \alpha$ is different from that just given in that the ranges of the functions f and g are α-regular.

<u>2.2. Theorem</u>.
For all admissible $\alpha > \omega$ there are α-r.e. sets $A,B \subseteq \alpha$ so that $A \leq_1 B$ and $B \leq_1 A$, but not $A \equiv B$.

Proof.

Let $\alpha > \omega$ be some fixed admissible ordinal. Fix a simultaneous α-recursive enumeration $\{h_e^\sigma\}_{\sigma<\alpha, e<\alpha^*}$ of the partial α-recursive 1-1 functions with domain and range subsets of α. (Such an enumeration exists by an appropriate application of Σ_1-Uniformisation for L_α to some universal α-recursive function.) We want to define α-recursive functions $f,g:\alpha \xrightarrow{1-1} \alpha$ and α-r.e. sets $A,B \subseteq \alpha$ such that the following requirements are satisfied:

(**) $A \equiv_1 B$ via f,g

(R_e) if h_e is total and onto, then $h_e[A] \neq B$, for all $e < \alpha^*$.

How can this be done? We choose functions f and g as follows:

$$f(\delta) = \delta + 1 \quad \text{for all} \quad \delta < \alpha$$
$$g(\delta+1) = \delta + 1 \quad \text{for all} \quad \delta < \alpha.$$

The values of g at limit ordinals are going to be determined during the construction. Assuming for a moment that g has already been completely defined, we observe that the orbits included by f and g can be sketched as follows:

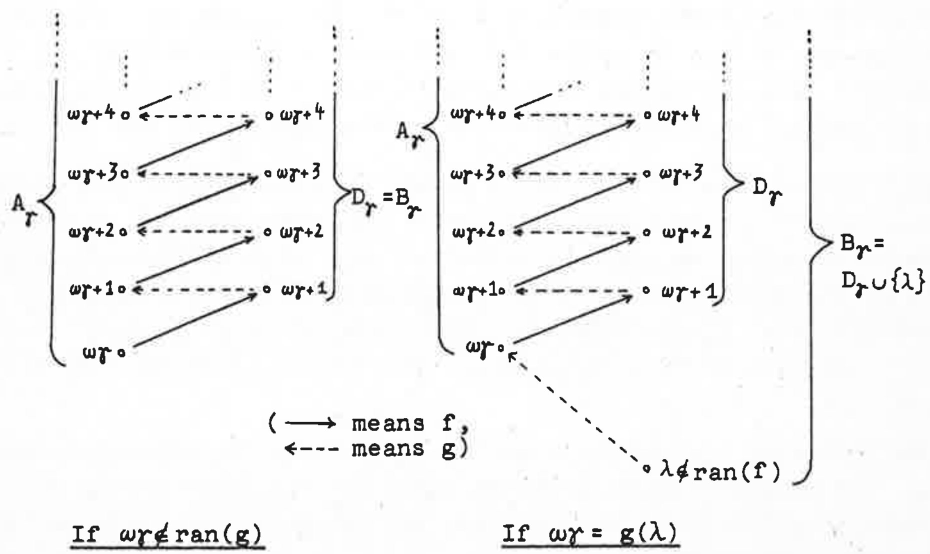

Figure 1

We define $A_\gamma := \{\omega\gamma + n \mid n < \omega\}$

$D_\gamma := \{\omega\gamma + n + 1 \mid n < \omega\}$

$$B_\gamma := \begin{cases} D_\gamma & \text{if } \omega\gamma \notin \mathrm{ran}(g) \\ D_\gamma \cup \{\lambda\} & \text{if } \omega\gamma = g(\lambda). \end{cases}$$

It is obvious that (A_γ, B_γ), $\gamma < \alpha$, are the orbits here.

The reouirememt (**) can now easily be satisfied: From (*) above we have for all sets $A, B \subseteq \alpha$:

$$A \equiv_1 B \text{ via } f,g \text{ iff}$$

$$(***) \qquad \forall_\gamma < \alpha((A_\gamma \subseteq A \text{ and } B_\gamma \subseteq B) \vee (A_\gamma \cap A = B_\gamma \cap B = \emptyset)).$$

Thus, in the construction to follow we ensure (**) by making (***) true:

> whenever a member of A_γ is to enter A or a member of B_γ is to enter B, put all elements of A_γ into A and all elements of B_γ into B.

How to make R_e true? Remember the discussion following the description of the proof of Myhill's Theorem in CRT above. f and g have been chosen so as to enable us to create deliberately the situation recognized above as hazardous to a proof of Myhill's Theorem for $\alpha > \omega$: wait until h_e has settled down on A_γ, then pick some $\lambda \notin h_e[A_\gamma]$ and define $g(\lambda) := \omega\gamma$, thus adding λ to B_γ. Then of course $h_e^{-1}(\lambda) \notin A_\gamma$. We have now the opportunity to achieve $h_e[A] \neq B$ by trying to ensure $h_e^{-1}(\lambda) \notin A \Longleftrightarrow \lambda \in B$, without hurting (***): if $h_e^{-1}(\lambda)$ is already in A, we keep all elements of A_γ outside A and all elements of B_γ outside B; if $h_e^{-1}(\lambda)$ is not yet in A, we put all elements of $A_\gamma(B_\gamma)$ into $A(B)$, and hope that $h_e^{-1}(\lambda)$ will stay out of A forever.

We want to describe the strategy for R_e, as it is used in the construction below. We may assume for this discussion that h_e is total and onto. We enumerate the pairs $\{(\lambda, g(\lambda)) \mid \lambda < \alpha \text{ limit}\}$ and the sets A and B in stages $\sigma < \alpha$.

Fix some stage σ_0. Assume that α-finite parts $A^{<\sigma_0}$ of A and $B^{<\sigma_0}$ of B have been enumerated and that an α-finite part of the set $\{(\lambda, g(\lambda)) : \lambda \text{ a limit}\}$ has been determined.

<u>Step 1</u>: Start an attempt for R_e:
Choose some γ which has not been mentioned in the construction so far. In particular, $A_\gamma \cap A^{\sigma_0} = \emptyset$, $D_\gamma \cap B^{\sigma_0} = \emptyset$, $\omega\gamma$ not yet in $\mathrm{ran}(g)$.

Step 2: Continue this attempt:
When some stage $\sigma_1 > \sigma_0$ is reached at which h_e has been enumerated so far that $\mathrm{dom}\left(h_e^{\sigma_1}\right) \supseteq A_\gamma$, we continue this attempt (such a stage must exist since A_γ is α-finite and h_e is total): Choose some $\lambda \notin h_e[A_\gamma]$ not yet in $\mathrm{dom}(g)$ and add the pair $(\lambda, \omega\gamma)$ to g at stage σ_1.

Step 3: Complete this attempt:
When some stage $\sigma_2 > \sigma_1$ is reached at which h_e has been enumerated so far that $\lambda \in \mathrm{ran}\left(h_e^{\sigma_2}\right)$, then we may complete this attempt (such a stage σ_2 exists since h_e is onto): By the construction we know that $h_e^{-1}(\lambda) \notin A_\gamma$. Hence $h_e^{-1}(\lambda) \in A_{\gamma'}$ for some $\gamma' \neq \gamma$. Let $A^{<\sigma_2}$ be the part of A enumerated so far.

Case 1: $A_{\gamma'} \subseteq A^{<\sigma_2}$. Do nothing.

Case 2: $A_{\gamma'} \cap A^{<\sigma_2} = \emptyset$. Then put all elements of A_γ into A and all elements of $B_\gamma = \{\lambda\} \cup D_\gamma$ into B at stage σ_2. (Only this action is called "completing this attempt.") Hope that the elements of $A_{\gamma'}$ will stay out of A forever.

Problems occur, of course, when one tries to treat all α-recursive permutations simultaneously. Conflicts between different α-recursive permutations may arise in Step 3 of the strategy: for the sake of some $h_{e'}$, $e' \neq e$, perhaps the elements of $A_{\gamma'}$ will be put into A later.

To solve such conflicts, the appropriate tool is the α-finite injury priority method (the CRT-version of this method was invented by Friedberg and Muchnik in 1956). One essential requirement for this method to work is satisfied: one may start an attempt at obtaining $h_e[A] \neq B$ unboundedly often. (The stage σ_0 in the sketch above was arbitrary.)

It is rather easy to see that for α with the property $\Delta_2\mathrm{cf}\alpha = \alpha$ (e.g., if α is a regular L-cardinal) a priority construction which uses $<$ on α as priority ordering, along the previous outline succeeds without complications.

To make the construction work for all α, we will have to adopt a technique for creating a priority ordering of length $\Delta_2\mathrm{cf}\alpha$. We use Shore Blocking here (Shore [21]), following the exposition of this method in Simpson [22]. Proofs of the following propositions concerning $\Delta_2\mathrm{cf}\alpha$ may be found there.

2.3. Definition.
$\Delta_2 \mathrm{cf}\alpha$ is the least $\delta \leq \alpha$ such that some $\Sigma_2(L_\alpha)$-function maps δ cofinally into α.
$\Delta_2 \mathrm{cf}\alpha^*$ is the least $\delta \leq \alpha^*$ such that some $\Sigma_2(L_\alpha)$-function maps δ cofinally into α^*.

2.4. Lemma.
(i) $\Delta_2 \mathrm{cf}\alpha = \Delta_2 \mathrm{cf}\alpha^*$
(ii) Let $\nu < \Delta_2 \mathrm{cf}\alpha$. If $\{I_\mu | \mu < \nu\}$ is a simultaneously α-r.e. sequence of α-finite sets I_μ, then $\bigcup\{I_\mu | \mu < \nu\}$ is α-finite.

2.5. Lemma.
There are an α-recursive function $\hat{H}: \alpha \times \Delta_2 \mathrm{cf}\alpha \longrightarrow \alpha^*$ and a $\Delta_2(L_\alpha)$-function $H: \Delta_2 \mathrm{cf}\alpha \longrightarrow \alpha^*$ such that

(i) H is nondecreasing and continuous; $H(0) = 0$; ran(H) is cofinal in α^*.
(ii) For each $\sigma < \alpha$ the function $\nu \mapsto \hat{H}(\sigma, \nu)$ is nondecreasing and continuous; $\hat{H}(\sigma, 0) = 0$, $\hat{H}(\sigma, \nu) \leq \sigma$.
(iii) For each $\nu < \Delta_2 \mathrm{cf}\alpha$ there is some $\sigma < \alpha$ such that $(\forall \mu \leq \nu)(\forall \tau \geq \sigma)(\hat{H}(\tau, \mu) = H(\mu))$.

2.6. Definition. (The change function).
We say that $H(\nu)$ changes its value at stage σ iff

$$\neg(\exists \tau < \sigma)(\forall \sigma')(\tau \leq \sigma' < \sigma \longrightarrow \hat{H}(\sigma', \mu) = \hat{H}(\sigma, \mu)) .$$

The change function ch gives the initial segment of α^* in which nothing changes:

$$\mathrm{ch}(\sigma) := \bigcup\{\hat{H}(\sigma, \nu) | \text{ for no } \mu \leq \nu \text{ does } H(\mu) \text{ change its value at stage } \sigma\}.$$

The essential property of ch is the second assertion of the following lemma. (The proof is trivial.)

2.7. Lemma.
(i) For all $e < \mathrm{ch}(\sigma)$ there is exactly one ν such that $\hat{H}(\sigma, \nu) \leq e < \hat{H}(\sigma, \nu+1)$; and neither $H(\nu)$ nor $H(\nu+1)$ change their value at stage σ.
(ii) If $H(\nu) \leq e < H(\nu+1)$, and the interval $[H(\nu), H(\nu+1)[$ reaches its final position at stage σ (i.e., σ is least such that for all $\tau \geq \sigma$ $\hat{H}(\tau, \nu) = H(\nu)$ and $\hat{H}(\tau, \nu+1) = H(\nu+1)$) then $\mathrm{ch}(\sigma) \leq e$.

We use $\hat{H}$ to give priorities $< \Delta_2 cf\alpha$ to the R_e's in the following manner: Asymptotically (as σ goes to α), the priority of $e < \alpha^*$ is $\nu < \Delta_2 cf\alpha$ if $H(\nu) \leq e < H(\nu+1)$. If e_i has priority ν_i $(i = 1,2)$, then e_1 has higher priority than e_2 iff $\nu_1 < \nu_2$. Since H is in general not α-recursive, we use $\hat{H}$ instead, and treat e as if it had priority ν at stage σ if $\hat{H}(\sigma,\nu) \leq e < \hat{H}(\sigma,\nu+1)$. Since for each $e < \alpha^*$ there is a stage σ_e such that for all $\sigma \geq \sigma_e$ holds $ch(\sigma) > e$, these "guessed" priorities are the true ones after boundedly many stages for each initial segment of α^*.

It is essential for the construction to work that a new attempt at satisfying R_e is started at stage σ if $e \in [H(\nu), H(\nu+1)$ and this interval reaches its final position at stage σ. This is what we use the change function for.

If $e_1, e_2 \in [\hat{H}(\sigma,\nu), \hat{H}(\sigma,\nu+1)[$, they are treated at stage σ as if they had the same priority. If a conflict between such e_1, e_2 occurs at stage σ, we give priority to that e_i which "comes first", i.e., for which an attempt is to be completed at stage σ. This causes no harm, since if an attempt for R_e is completed at stage σ, this attempt is not injured at a later stage for the sake of an e' of the same priority (cf. the proofs of 2.9 and 2.10).

As last of our preliminaries, we choose some α-recursive partition $\{Z_e\}_{e<\alpha^*}$ of α. (E.g., let $Z_e = \{\langle e,\delta\rangle : \delta < \alpha\}$ for some fixed α-recursive bijection $\langle\ ,\ \rangle : \alpha^* \times \alpha \xrightarrow[\text{onto}]{1-1} \alpha$.) Only A_γ with $\gamma < Z_e$ will be used in the strategy for R_e as described above.

<u>The construction</u>

By simultaneous recursion on $\sigma < \alpha$ we enumerate sets $A, B \subseteq \alpha$, and define $g \restriction \{\lambda \mid \lambda < \alpha \text{ limit}\}$. Let $A^{<\sigma}$ and $B^{<\sigma}$ be the parts of A and B respectively enumerated before stage σ.

<u>Stage σ</u>.

<u>Injuries caused by changes of H</u>:

All attempts for R_e such that $e \geq ch(\sigma)$ are <u>injured</u> now.

<u>Strategy for</u> R_e:

Determine the unique $e < \alpha^*$ such that $\sigma \in Z_e$. Consider three cases:

<u>Case 1</u>. (corresponds to Step 1 above).
If all attempts for R_e started before stage σ have been injured in the meantime, then <u>start an attempt for</u> R_e as follows:

Choose some witness $\gamma \in Z_e$ so large that no member of A_γ or D_γ has occured in the construction so far.

Case 2. (corresponds to Step 2).
If at some stage $\sigma_0 < \sigma$ an attempt for R_e has been started and not been injured in the meantime, which used γ as witness, and $A_\gamma \subseteq h_e^\sigma$, and $\omega\gamma$ is not yet in ran(g), then continue this attempt as follows: Choose some limit $\lambda = \omega\xi$, $\xi \in Z_e$, which has not occured in the construction before (in particular $\lambda \notin h_e[A_\gamma]$). Add the pair $(\lambda, \omega\gamma)$ to graph (g) now.

Case 3. (corresponds to Step 3).
If an attempt for R_e has been started at some stage σ_0, using γ as witness, such that

(i) it has been continued at some stage $\sigma_1 > \sigma_0$, by putting $(\lambda, \omega\gamma)$ into graph (g).

(ii) $\lambda \in \mathrm{ran}(h_e^\sigma)$ for λ as in (i).

(iii) this attempt has not been injured nor has it been completed in the meantime.

(iv) $h_e^{-1}(\lambda) \notin A^{<\sigma}$,

then complete this attempt as follows:

(1) Enumerate all members of A_γ into A and all members of $B_\gamma = \{\lambda\} \cup D_\gamma$ into B now.

(2) All attempts concerning only e' with lower priority than e are injured now. (These are the ordinals $e' < \alpha^*$ so that $e < \hat{H}(\sigma,\nu) \leq e'$ for some $\nu < \Delta_2 \mathrm{cf}\alpha$.)

(3) Compute γ' such that $h_e^{-1}(\lambda) \in A_{\gamma'}$, and e' such that $\gamma' \in Z_{e'}$. If e' has the same priority e (i.e., $\hat{H}(\sigma,\nu) \leq e$, $e' < \hat{H}(\sigma,\nu+1)$ for some $\nu < \Delta_2 \mathrm{cf}\alpha$) and the current attempt for $R_{e'}$, if any is going on, uses γ' as witness, then this attempt is injured now. (Observe for later use that in this case holds $A_{\gamma'} \cap A^{<\sigma} = \emptyset$, by (iv) above.)

Make sure that g will be total:
If $\omega\sigma$ is not yet in dom(g) then let

$$g(\omega\sigma) := \omega \cdot (\text{the } \sigma\text{-th members of } Z_0).$$

(We can safely assume that $h_0 = \emptyset$.)

2.8. Lemma.

(i) g is α-recursive, total, and one-one.

(ii) A and B are α-r.e. subsets of α.

(iii) $A \equiv_1 B$ via f,g.

Proof.

(i) and (ii) follow immediately from the construction. Injectivity of g is guaranteed by the fact that each attempt for any R_e is continued at most once.

(iii) Obviously, by Step 3 in the construction, condition (***) is satisfied.

2.9. The Injury Lemma.

For all $\nu < \Delta_2 cf\alpha$ holds:

(i) The injury set

$$I_\nu := \{\sigma < \nu \mid \text{at stage } \sigma \text{ an attempt for some } R_e \text{ with } e < \hat{H}(\sigma,\nu+1) \text{ is injured}\}$$

is α-finite.

(ii) The completion set

$$C_\nu := \{\sigma < \alpha \mid \text{at stage } \sigma \text{ an attempt for some } R_e \text{ with } e < \hat{H}(\sigma,\nu+1) \text{ is completed}\}$$

is α-finite.

Proof.

(i) and (ii) are proved simultaneously by induction on $\nu < \Delta_2 cf\alpha$.

So, let such a ν be given, and assume by the induction hypothesis that the sets I_μ and C_μ are α-finite for all $\mu < \nu$. Of course, the families $\{I_\mu \mid \mu < \nu\}$ and $\{C_\mu \mid \mu < \nu\}$ are simultaneously α-r.e., so by Lemma 2.4. (ii) the set $\bigcup\{C_\mu \mid \mu < \nu\}$ is α-finite. Choose a strict upper bound $\sigma_0 < \alpha$ of this set. From Lemma 2.5. (iii) and the definition of the change function it follows that we can choose some $\sigma_1 > \sigma_0$ such that:

$$ch(\sigma) \geq \hat{H}(\sigma,\nu+1) = H(\nu+1) \quad \text{for all } \sigma > \sigma_1,$$

hence

(1) $\quad (\forall\sigma > \sigma_1)(\forall e < H(\nu+1)) \quad (e < ch(\sigma))$.

It is clear from the definition of σ_0 that

(2) $\quad (\forall\sigma > \sigma_1)(\forall e < H(\nu+1))$ (no attempt for some R_{e}' with higher priority than R_e is completed at stage σ).

Next we define an α-r.e. set $b \subseteq H(\nu+1) \times \alpha$:

$$b := \{(e,\sigma) \mid \sigma > \sigma_1 \text{ and an attempt for } R_e \text{ is completed at stage } \sigma \text{ and } H(\nu) \leq e < H(\nu+1)\}.$$

We show that b is a function. So assume $(e,\sigma_e) \in b$. Now the attempt for R_e which is completed at stage σ_e can not be injured after stage σ_e. ($ch(\sigma) > e$, since we are beyond σ_1. No e' with higher priority can injure e now, since we are beyond σ_0. No e' with the same priority can injure this attempt for R_e now, since $A_\gamma \subseteq A$ then; cf. Case 3 in the construction.) Hence never a new attempt for R_e is started after σ_e. Thus for no $\sigma > \sigma_e$ can hold $(e,\sigma_e) \in b$. dom(b) is an α-finite set by Σ_1-separation below α^*; so b and ran(b) are α-finite as well by admissibility. Choose a strict upper bound $\sigma_2 > \sigma_1$ of ran(b). By (1), (2), and the definition of b we conclude that no attempt for any $e < H(\nu+1)$ can be injured after stage σ_2. Thus $\sigma_2 < \alpha$ is an upper bound of I_ν. Since I_ν is clearly α-recursive, it is α-finite by Δ_1-separation. Thus (1) is proved.

Next we must show that C_ν is α-finite as well. Since C_ν is α-recursive, it suffices to show that $C_\nu \cap \{\sigma < \alpha \mid \sigma > \sigma_2\}$ is bounded. It is clear by the definition of σ_0 and σ_1 that this set equals

$$\{\sigma > \sigma_2 \mid \text{an attempt for some } R_e \text{ so that } H(\nu) \leq e < H(\nu+1) \text{ is completed at stage } \sigma\}.$$

But for each $e \in [H(\nu),H(\nu+1)[$ at most one attempt is completed after stage σ_2. Arguing as above for the set b, we see that the α-r.e. set

$$\{(e,\sigma) \mid \sigma > \sigma_2 \text{ and } e < H(\nu+1) \text{ and an attempt for } R_e \text{ is completed at stage } \sigma\}$$

is a function with domain bounded below α^*, hence is α-finite. So its range is α-finite, as was to be shown.

<u>2.10. Lemma</u>.

For each $e < \alpha$ the requirement R_e is satisfied.

<u>Proof</u>.

Let e be such that h_e is total and onto. Fix the unique $\nu < \Delta_2 cf\alpha$ such that $H(\nu) \leq e < H(\nu+1)$. From the Injury Lemma 2.9 it follows that there is some stage after which no attempt for R_e is injured. Hence there must be a last attempt for R_e. (Z_e is unbounded!). We show that it succeeds. Let σ_0 be the stage at which this last attempt for R_e is started. By the construction, $e < ch(\sigma)$ for all $\sigma > \sigma_0$.

Hence $H(\nu)$ and $H(\nu+1)$ don't ever change their value after stage σ_0. I.e., that the sets $\{e' \mid e'$ has lower (higher) priority than e at stage $\sigma\}$ are independent of σ for $\sigma > \sigma_0$. Since h_e is total, and A_γ is α-finite, there is some stage $> \sigma_0$ after which $\mathrm{dom}(h_e^\sigma) \supseteq A_\gamma$. So there is a stage $\sigma_1 > \sigma_0$ such that this attempt for R_e is continued (again because Z_e is unbounded). So we have $g(\lambda) = \omega\gamma$ for some $\lambda \notin h_e[A_\gamma]$. Since h_e is onto, there is some stage $> \sigma_1$ at which λ enters $\mathrm{ran}(h_e)$. Let σ_2 be the least stage in Z_e after that. Consider two cases:

Case 1. $h_e^{-1}(\lambda) \in A^{<\sigma_2}$. Then all elements of B_γ stay outside B forever by the construction. So $\lambda \notin B$ but $h_e^{-1}(\lambda) \in A$, hence $h_e[A] \neq B$.

Case 2. $h_e^{-1}[\lambda] \notin A^{<\sigma_2}$. Then by the construction, all elements of B_γ are put into B at stage σ_2. Hence $\lambda \in B$. We must show that $h_e^{-1}(\lambda) \notin A$. There is a unique γ' such that $h_e^{-1}(\lambda) \in A_{\gamma'}$. Fix the unique e' such that $\gamma' \in Z_{e'}$. If e' has lower priority than e at stage σ_2, the current attempt for e' is injured; since attempts for e' started later must use some new "γ" as witness, the members of $A_{\gamma'}$ are never put into A after stage σ_2. If e' has higher priority than e at stage σ_2, then no attempt for e' can be completed after stage σ_2 (otherwise the last attempt for R_e would be injured then, contradiction), hence $A_{\gamma'} \cap A = \emptyset$ remains true. If e' has the same priority as e at stage σ_2, and the current attempt for e' has γ' as its witness, then this is injured at stage σ_2 (Case 3 (3) in the construction). If e' does not use γ' in a current attempt, γ' will never be chosen as witness for some attempt for $R_{e'}$, since it is not "new".

This finishes the proof of Theorem 2.2.

If we look at Theorem 2.2 from a β-recursionist's point of view, we get

2.11. Theorem.
Myhill's Theorem fails for all limit ordinals β with $\sigma\mathrm{lcf}\beta > \omega$.

Proof.
Case 1. β is weakly admissible, i.e., $\beta^* \leq \sigma\mathrm{lcf}\beta$. In this case, we can reduce the proposition for β (there are β-r.e. sets $A,B \subseteq \beta$ so that $A \equiv_1 B$ but not $A \equiv B$) to the same proposition for an admissible structure $\langle L_\alpha,\in,T\rangle$, where $\alpha = \sigma\mathrm{lcf}\beta$, $T \subseteq \alpha$ is an α-regular set which codes the Δ_0-satisfaction relation of L_β. $\mathfrak{A}$ is the admissible

collapse of L_β, as defined in [13]. The proof of Theorem 2.2. works equally well for $\mathfrak{A}$. It is easily seen that the counterexample for $\mathfrak{A}$ obtained in this eay can be transformed into a counterexample for L_β by the inverse of the collapsing function.

Case 2. β is strongly inadmissible, i.e., $\beta^* > \sigma lcf\beta > \omega$. In [4] it is shown that there are sets $A,B \subseteq \beta$ and β-recursive 1-1 functions $f,g:\beta \longrightarrow \beta$ so that $A \equiv_1 B$ via f,g, but A,B are not β-recursively isomorphic. The proof uses an enumeration of the Gödel Numbers of the β-recursive permutations of L_β, which is, of course, not a β-r.e. set. (But it is easily seen that A,B can be chosen so as to be definable over L_β.)

§3. MYHILL'S THEOREM HOLDS IF $\sigma lcf\beta = \omega$

In §2, we disproved Myhill's Theorem for all β with $\sigma lcf\beta > \omega$. How is the situation if $\sigma lcf\beta = \omega$? We know that for $\beta = \omega$ the theorem holds. If $\hat{\beta} = \omega$, Myhill's original proof works just as well. But even for arbitrary limit ordinals β with $\sigma lcf\beta = \omega$ the theorem is true:

3.1. Theorem.
Let $\sigma lcf\beta = \omega$. Then $A \equiv_1 B \Rightarrow A \equiv B$, for all $A,B \subseteq \beta$.

Proof.
Let $A,B \subseteq \beta$ and β-recursive functions $f,g:\beta \xrightarrow{1-1} \beta$ be given so that $A \equiv_1 B$ via f,g. We use the central idea of Myhill's proof (as recalled at the beginning of §2). The construction of a β-recursive isomorphism h between A and B is carried out in ω stages, and thus the growth of dom(h) and ran(h) during the construction can be controlled in such a way that at each stage n, if x is a candidate to enter dom(h), we can guarantee that at some stage $m \geq n$ a possible image for x under h is available. (Recalling the definition in §2 of the orbits induced by f and g we remark that this must be an element of $[f(x)]^B$ not yet in ran(h).)

We shall define a β-recursive function h and show in a series of lemmas that h is an isomorphism between A and B, i.e., h is total, onto, one-one, and $h[A] = B$. The definition of h will involve f and g only, A and B are not mentioned. Recalling statement (*) of §2 we see that to achieve $h[A] = B$ we have to define h in such a way that

$$h \upharpoonright [x]^A \text{ maps } [x]^A \text{ one-one onto } [f(x)]^B, \text{ for all } x \in \beta.$$

The problem with this aim is that the orbits cannot be dealt with in a β-recursive way. (E.g., the questions if $[x]^A \subseteq \mathrm{ran}(g)$, or if $[x]^A$ is finite or infinite, are not β-recursively decidable.) So we have to use approximations to the orbits.

Since $\sigma \mathrm{lcf}\beta = \omega$, there are two $\Sigma_1(L_\beta)$-sequences $\langle f_n \mid n \in \omega\rangle$ and $\langle g_n \mid n \in \omega\rangle$, $f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$, and $g_0 \subseteq g_1 \subseteq g_2 \subseteq \ldots$, such that

$$f = \bigcup\{f_n \mid n \in \omega\} \quad \text{and} \quad g = \bigcup\{g_n \mid n \in \omega\}.$$

If $\beta^* = \omega$, we can additionally assume that $|f_n| \leq n$ and $|g_n| \leq n$ for all $n \in \omega$. (If necessary, take some β-recursive function $r: \omega \xrightarrow[\text{onto}]{1-1} \beta$, and replace f_n, g_n by $f_n \upharpoonright r[n]$, $g_n \upharpoonright r[n]$ respectively.)

3.2. Definition.

Let $n \in \omega$. $((g_n f_n)^0 := (f_n g_n)^0 := \mathrm{id}_\beta.)$

For $x, x', y, y' \in \beta$ we define:

$$x \sim_n^A x' :\Leftrightarrow (\exists k \in \omega)(x = (g_n f_n)^k(x') \vee x' = (g_n f_n)^k(x))$$

$$y \sim_n^B y' :\Leftrightarrow (\exists k \in \omega)(y = (f_n g_n)^k(y') \vee y' = (f_n g_n)^k(y))$$

$$[x]_n^A := \{x' \mid x' \sim_n^A x\}$$

$$[y]_n^B := \{y' \mid y' \sim_n^B y\}$$

A pair $([x]_n^A, [y]_n^B)$ of equivalence classes is called an n-orbit (w.r.t. $\langle f_n \mid n \in \omega\rangle$, $\langle g_n \mid n \in \omega\rangle$) if and only if

$$(\exists k \in \omega)(x = (g_n f_n)^k g_n(y) \vee y = (f_n g_n)^k f_n(x)) .$$

3.3. Lemma.

(1) $\sim_n^A$ and $\sim_n^B$ are equivalence relations on β which are refinements of $\sim^A$ and $\sim^B$, respectively.

(2) Let $x, y \in \beta$.
$([x]^A, [y]^B)$ is an orbit if and only if

$$(\exists n \in \omega)(([x]_n^A, [y]_n^B) \text{ is an n-orbit}).$$

$(\exists \bar{y} \in \beta)(([x]_n^A, [\bar{y}]_n^B))$ is an n-orbit if and only if

$$x \in \mathrm{dom}(f_n) \cup \mathrm{ran}(g_n).$$

$(\exists \bar{x} \in \beta)(([\bar{x}]_n^A, [y]_n^B)$ is an n-orbit) if and only if

$$y \in \text{dom}(g_n) \cup \text{ran}(f_n).$$

(3) $\left.\begin{array}{l} x \sim_n^A x' \\ y \sim_n^B y' \end{array}\right\}$ are β-recursive relations of $\left\{\begin{array}{l} n,x,x' \\ n,y,y' \end{array}\right.$

(4) The mappings $(n,x) \mapsto [x]_n^A$ and $(n,y) \mapsto [y]_n^B$ are β-recursive.

(5) For each $n \in \omega$, the set of n-orbits is a partial one-one mapping.

Proof.
Trivial.

We shall need a kind of linear ordering (of order type $\leq \omega$) of each class $[x]_n^A$ and $[y]_n^B$. For this sake, we first choose a distinguished representative in each equivalence class and then define some notion of "distance from the distinguished element."

3.4. Definition.
Let $n \in \omega$.

$m_n^A(x) := \min([x]_n^A)$, for $x \in \beta$

$m_n^B(y) := \min([y]_n^B)$, for $x \in \beta$

$ds_n^A(x) :=$

$$\begin{cases} 0 \;, & \text{if } x = m_n^A(x) \\ 2k \;, & \text{if } k > 0 \text{ and } x \sim (g_n f_n)^k(m_n^A(x)) \text{ and} \\ & \text{and } (\forall j)(0 \leq j < k \to x \not\sim (g_n f_n)^j(m_n^A(x))) \\ 2k-1 \;, & \text{if } k > 0 \text{ and } x \sim (g_n f_n)^{-k}(m_n^A(x)) \text{ and} \\ & \text{and } \neg(\exists j > 0)(x \sim (g_n f_n)^j(m_n^A(x))) \;. \end{cases}$$

$ds_n^B(y)$ is defined analogously (exchange f_n and g_n, replace A by B and x by y).

3.5. Remark.
To make the meaning of ds_n^A , ds_n^B clearer, we consider two cases. Let $x = m_n^A(x)$.

Case 1. $[x]_n^A$ is "cyclic", i.e., $(\exists j > 0)((g_n f_n)^j(x) = x)$. Let j_0 be the least such j. Then

$$[x]^A = [x]_n^A = \{(g_n f_n)^k(x) \mid 0 \leq k < j_0\}, \quad \text{and}$$

$$ds_n^A((g_n f_n)^k(x)) = 2k \quad \text{for all} \quad k,\ 0 \leq k < j_0.$$

Case 2. $[x]_n^A$ is not "cyclic."
Then for each $x' \in [x]_n^A$ there is exactly one integer z such that

$$x' = (g_n f_n)^z(x) \quad ; \quad \text{and}$$

$$ds_n^A(x') = 2z \quad , \quad \text{if} \quad z \geq 0$$

$$ds_n^A(x') = -2z-1 \ , \ \text{if} \quad z < 0.$$

3.6. Lemma.

(1) $m_n^A(x)$ is a β-recursive function of n and x;
$x \sim_n^A x'$ if and only if $m_n^A(x) = m_n^A(x')$.
Analogously for m_n^B.

(2) $ds_n^A(x)$ and $ds_n^B(y)$ are β-recursive functions of n,x (respectively n,y).

(3) For all $n \in \omega$ and all $x,y \in \beta$ are $ds_n^A \restriction [x]_n^A$ and $ds_n^A \restriction [y]_n^B$ one-one functions.

Proof.
Trivial.

3.7. Construction of h.
By induction on $n \in \omega$ an $\Sigma_1(L_\beta)$-sequence $\langle h_n \mid n \in \omega\rangle$ of partial mappings $h_n \in L_\beta$ is defined. h is obtained as the union $\bigcup\{h_n \mid n \in \omega\}$. (Note that $h_n \cap h_m = \emptyset$ for $n \neq m$).
Abbreviation: $h_{<n} := \bigcup\{h_m \mid m < n\}$.

$$h_n := \{(x,y) \in \beta^2 \mid x,y \text{ satisfy (1) and (2) and (3)}\}$$

where (1), (2) and (3) are the following conditions:

(1) $([x]_n^A, [y]_n^B)$ is an n-orbit and $x \notin \text{dom}(h_{<n})$ and $y \notin \text{ran}(h_{<n})$

(2) $(\exists j > 0)(x \sim (g_n f_n)^j(x)) \vee$
$\vee\ (|[x]_n^A - \text{dom}(h_{<n})| \geq 2$ and $|[y]_n^B - \text{ran}(h_{<n})| \geq 2)$

(3) $(\forall x')(x' \in [x]_n^A - \mathrm{dom}(h_{<n}) \longrightarrow ds_n^A(x') \geq ds_n^A(x))$ and

$(\forall y')(y' \in [y]_n^B - \mathrm{ran}(h_{<n}) \longrightarrow ds_n^B(y') \geq ds_n^B(y))$.

3.8. Remark.

This construction may seem to be too involved. Why not add as many pairs $(x',y') \in [x]_n^A \times [y]_n^B$ to h at stage n as possible, if $([x]_n^A, [y]_n^B)$ is an n-orbit? At the first glance, this strategy would perhaps make h total and onto. But, then it could happen that at some stage n e.g., $[x]^A \subseteq \mathrm{dom}(h_{<n})$, and $[x]_n^B$ gets a new element at stage $n+1$. Then we could never make h onto. The solution to this problem is as follows: At most one element of $[x]_n^A$ is allowed to enter $\mathrm{dom}(h)$ at stage n. Condition (2) then will ensure that

$$[x]_n^A - \mathrm{dom}(h_{<n+1}) \neq \emptyset \quad \text{or} \quad [x]^A = [x]_n^A \text{ is finite.}$$

This will suffice to guarantee that for all $n \in \omega$

$$\text{if } [x]^A \text{ is infinite, then } [x]^A - \mathrm{dom}(h_{<n+1}) \text{ is infinite.}$$

So the construction never breaks down for lack of suitable elements in $[x]^A - \mathrm{dom}(h_{<n})$. (The strategy concerning the y's is the same.)

It just remains to make h total and onto. For this purpose, the first elements of $[x]_n^A - \mathrm{dom}(h_{<n})$, $[y]_n^B - \mathrm{ran}(h_{<n})$ (with respect to the distance functions ds_n^A and ds_n^B) are chosen to enter $\mathrm{dom}(h_n)$, $\mathrm{ran}(h_n)$, if any are. For this idea to work, the distance functions must finally "settle down", i.e., $ds_n^A(x)$ must be constant for n large enough.

The details are given in the following lemmas.

3.9. Lemma.

Let $x,y \in \beta$.

(1) For all $n \in \omega$ holds: $[x]_n^A \subseteq [x]_{n+1}^A$ and $[y]_n^B \subseteq [y]_{n+1}^B$.

$$[x]^A = \bigcup\{[x]_n^A \mid n \in \omega\} \quad \text{and} \quad [y]^B = \bigcup\{[y]_n^B \mid n \in \omega\}.$$

(2) $\lim_{n\to\infty} m_n^A(x) = \min([x]^A)$ and $\lim_{n\to\infty} m_n^B(y) = \min([y]^B)$.

(3) $\lim_{n\to\infty} ds_n^A(x)$ exists and $\lim_{n\to\infty} ds_n^B(y)$ exists.

(4) The mappings $x \mapsto \lim_{n\to\infty} ds_n^A(x)$ and $y \mapsto \lim_{n\to\infty} ds_n^B(y)$ are one-one on the respective equivalence classes, e.g., if $x \sim^A x'$ and $\lim_{n\to\infty} ds_n^A(x') = \lim_{n\to\infty} ds_n^A(x)$, then $x = x'$.

(The limits are all in the discrete topology on β respectively ω.)

Proof.

Straightforward (use 3.6.).

3.10. Lemma.

(1) h is a partial one-one function: For all $x,x',y,y' \in \beta$ holds that if $(x,y),(x',y') \in h$, then $x = x' \Longleftrightarrow y = y'$.

(2) If $h(x) \simeq y$, then $y \sim^B f(x)$ (i.e., $([x]^A,[y]^B)$ is an orbit.)

Proof.

(1) Let $n \geq m$, $(x,y) \in h_m$, $(x',y') \in h_n$.

Assume first that $x = x'$. Then n must be equal to m. (If $n > m$, then $x \in dom(h_{<n})$, hence $x \notin dom(h_n)$ by 3.7.(1).) Since $([x]_n^A,[y]_n^B)$ and $([x]_n^A,[y']_n^B)$ are both n-orbits, by 3.3.(5) it follows that $[y]_n^B = [y']_n^B$. y and y' both satisfy 3.7.(3) and hence are identical.

The other direction is proved similarly.

(2) If $(x,y) \in h_n$, then $([x]_n^A,[y]_n^B)$ is an n-orbit, hence $([x]^A,[y]^B)$ is an orbit by 3.3.(2).

3.11. Lemma.

If $x \in \beta$ and $(gf)^k(x) = x$ for some $k > 0$, then $x \in dom(h)$.

If $y \in \beta$ and $(fg)^k(y) = y$ for some $k > 0$, then $y \in ran(h)$.

Proof.

We prove the first assertion by induction on $\lim_{n\to\infty} ds_n^A(x)$. Since $(gf)^k(x) = x$ for $k > 0$, both $[x]^A$ and $[f(x)]^B$ are finite (and have the same cardinality).

Choose m so large that $f \restriction [x]^A \subseteq f_m$ and $g \restriction [f(x)]^B \subseteq g_m$. Then for all $n \geq m$ holds:

$$[x]^A = [x]_n^A \text{ and } [f(x)]^B = [f(x)]_n^B \text{ and}$$

$$([x]_n^A, [f(x)]_n^B) \text{ is an } n\text{-orbit and}$$

$$\text{for all } x' \in [x]^A \text{ holds } ds_n^A(x') = ds_m^A(x')$$

(cf. 3.9.(3)).

By the induction hypothesis choose $n \geq m$ so large that

$$(\forall x' \in [x]^A)(ds_m^A(x') < ds_m^A(x) \longrightarrow x' \in dom(h_{<n})).$$

If $x \in dom(h_{<n})$, we are done. If not, then

$$[f(x)]_n^B - ran(h_{<n}) \neq \emptyset,$$

too, since $|[x]_n^A| = |[f(x)]_n^B|$, and $h_{<n}$ is one-one, and $h_{<n}^{-1}[[x]_n^B] \subseteq [x]_n^A$ by 3.10.

This means that 3.7.(1), (2), and (3) is satisfied for x and some $y \in [f(x)]^B$. (Note that $(fg)^k(y) = y$ for all $y \in [f(x)]^B$.) So $x \in dom(h_n)$ by the construction. The assertion concerning ran(h) is proved similarly.

<u>3.12. Lemma</u>.
Let $x,y \in \beta$ be such that $([x]^A,[y]^B)$ is an orbit. Assume that $(gf)^k(x) \neq x$ for all $k > 0$. Then the following assertions hold:

(1) For all $x' \in [x]^A$ and all $k > 0$ is $(gf)^k(x') \neq x'$, for all $y' \in [y]^B$ and all $k > 0$ is $(fg)^k(y') \neq y'$.

(2) If $([x]_n^A,[y]_n^B)$ is an n-orbit, then

$$|[x]_n^A \times [y]_n^B \cap h_n| = |[x]_n^A \cap dom(h_n)| = |[y]_n^B \cap ran(h_n)| \in \{0,1\}.$$

(3) $[x]_n^A - dom(h_{<n+1}) \neq \emptyset$ and $[y]_n^B - ran(h_{<n+1}) \neq \emptyset$, all $n \in \omega$.

(4) If $([x]_n^A,[y]_n^B)$ is an n-orbit, and $[x]_n^A$ is infinite, then $[x]_n^A - dom(h_{<n+1})$ and $[y]_n^B - ran(h_{<n+1})$ are both infinite.

<u>Proof</u>.
(1) is trivial, and (2) follows immediately from the construction 3.7.
(3) By induction on n:

<u>Case 1</u>. $[x]_n^A \cap dom(h_n) \neq \emptyset$.
Then $x \in dom(f_n) \cup ran(g_n)$, by the construction and 3.3.(2).
By (2) there is exactly one $x' \in [x]_n^A \cap dom(h_n)$.
By (1) it follows that $(gf)^k(x') \neq x'$ for all $k > 0$.
Hence by 3.7.(2)

$$[x']_n^A - dom(h_{<n})| \geq 2.$$

Therefore

$$[x]_n^A - dom(h_{<n+1}) = ([x']_n^A - dom(h_{<n})) - dom(h_n) \neq \emptyset.$$

Case 2. $[x]^A_n \cap \mathrm{dom}(h_n) = \emptyset$.

If $n = 0$, then $h_{<n+1} = h_0 = h_n$, hence $x \in [x]^A_n\text{-dom}(h_{<n+1})$.
If $n > 0$, then by the induction hypothesis is

$[x]^A_{n-1}\text{-dom}(h_{<n}) \neq \emptyset$, hence $[x]^A_n\text{-dom}(h_{<n+1}) \neq \emptyset$.

(4) By induction on n:
If $n = 0$ and $[x]^A_n$ is infinite, then $[x]^A_0\text{-dom}(h_0)$ is infinite by (2).
Now let $n > 0$. By (2), it suffices to prove that $[x]^A_n\text{-dom}(h_{<n})$ is infinite.

Case 1. For some $x' \in [x]^A_n$ is $[x']^A_{n-1}$ infinite.

By the induction hypothesis is $[x']^A_{n-1}\text{-dom}(h_{<n})$ infinite; hence $[x]^A_n\text{-dom}(h_{<n})$ is infinite (3.9.(1)).

Case 2. There are infinitely many (pairwise disjoint!) classes $[x']^A_{n-1} \subseteq [x]^A_n$.

By (3) for all these classes holds

$$[x']^A_{n-1}\text{-dom}(h_{<n}) \neq \emptyset;$$

hence $[x]^A_n\text{-dom}(h_{<n})$ is infinite.

(The proofs for $\mathrm{ran}(h)$ are the same.)

3.13. Lemma.
Let $([x]^A, [y]^B)$ be an orbit. Assume that $(gf)^k(x) \neq x$ for all $k > 0$. Then $x \in \mathrm{dom}(h)$ and $y \in \mathrm{ran}(h)$.

Proof.
We prove by induction on $\lim_{n\to\infty} ds^A_n(x)$ that $x \in \mathrm{dom}(h)$. (The proof of "$y \in \mathrm{ran}(h)$" is similar.)

By 3.9.(4) we know that the set

$$D := \left\{x' \in [x]^A \mid \lim_{n\to\infty} ds^A_n(x') < \lim_{n\to\infty} ds^A_n(x)\right\}$$

is finite. The induction hypothesis tells us that $D \subseteq \mathrm{dom}(h)$. Choose N so large that

$$\min([x]^A) \in [x]^A_N \text{ and } D \subseteq [x]^A_N \text{ and } D \subseteq \mathrm{dom}(h_{<N}) \text{ and}$$
$$\text{and } ([x]^A_N, [y]^B_N) \text{ is an } n\text{-orbit.}$$

The choice of N immediately implies that

$$ds_m^A(x') = \lim_{n\to\infty} ds_n^A(x') \quad \text{for all} \quad x' \in [x]_m^A,\ m \geq N,$$

in particular

$$ds_m^A(x') < ds_m^A(x) \quad \text{iff} \quad x' \in D \quad (\text{for all} \quad x' \in [x]_m^A).$$

If we can show that for some $m \geq N$ holds

$$[x]_m^A \cap \mathrm{dom}(h_m) \neq \emptyset,$$

then by the construction (3.7.(3)) either $x \in \mathrm{dom}(h_{<m})$ or $x \in \mathrm{dom}(h_m)$, and we are done. Consider two cases:

<u>Case 1</u>. $[x]_m^A$ is infinite for some $m \geq N$.

By 3.12., both $[x]_m^A - \mathrm{dom}(h_{<m})$ and $[y]_m^B - \mathrm{ran}(h_{<m})$ are infinite. Hence 3.7.(1), (2), and (3) is satisfied for some $x' \in [x]_m^A$, $y' \in [y]_m^B$. So $[x]_m^A \cap \mathrm{dom}(h_m) \neq \emptyset$.

<u>Case 2</u>. $[x]_m^A$ is finite for all $m \geq N$.

By 3.12.(3) we know that there are elements

$$x_1 \in [x]_N^A - \mathrm{dom}(h_{<N}) \quad \text{and} \quad y_1 \in [y]_N^B - \mathrm{ran}(h_{<N}).$$

Since $[x]^A$ is infinite, for some stage $m_A > N$ holds

$$[x]_{m_A}^A - [x]_{m_A-1}^A \neq \emptyset, \text{ i.e.,}$$

there is an $x'' \in [x]_{m_A}^A$ such that $[x'']_{m_A-1}^A \cap [x]_{m_A-1}^A = \emptyset$.

By 3.12.(1) and (3) there exists an element

$$x_2 \in [x'']_{m_A-1}^A - \mathrm{dom}(h_{<m_A}).$$

Analogously, we find $m_B > N$ and an element

$$y_2 \in ([y]_{m_B}^B - \mathrm{dom}(h_{<m_B})) - [y]_{m_B-1}^B.$$

Consider $m := \max(m_A, m_B)$.
Either for some stage n, $N \leq n < m$, holds

$$([x]_n^A \times [y]_n^B) \cap h_n \neq \emptyset,$$

or

$$\{x_1, x_2\} \subseteq [x]_m^A - \mathrm{dom}(h_{<m}) \text{ and } \{y_1, y_2\} \subseteq [y]_m^B - \mathrm{ran}(h_{<m}).$$

Since $x_1 \neq x_2$ and $y_1 \neq y_2$, in the latter case 3.7.(1), (2), and (3) is satisfied for some $x' \in [x]_m^A$, $y' \in [y]_m^B$ at stage m.

Hence $[x]^A_m \times [y]^B_m \cap h_m \neq \emptyset$.

This finishes the proof of Theorem 3.1.

§4. A FEW FURTHER RESULTS ON STRONG REDUCIBILITIES IN β-RECURSION THEORY.

As we have seen in §2, when one defines β-recursive isomorphism classes and $\equiv_1$-classes of subsets of β, then these notions differ (in some important cases even for β-recursive sets), whereas in CRT they coincide. Therefore we consider here some stronger notion of reducibility, namely $\leq^r_1$ (see Rogers [16]). (The definition was given in §1.) $\leq^r_1$ induces an equivalence relation $\equiv^r_1$, and gives rise to the following pleasing result, whose proof uses as main idea the proof of the Cantor-Schröder-Bernstein Theorem from set theory.

4.1. Theorem.

Let β be any limit ordinal. Then $A \equiv^r_1 B \Longrightarrow A \equiv B$ for all $A, B \subseteq \beta$.

Proof.

If $\sigma\mathrm{lcf}\beta = \omega$, we use Theorem 3.1. So let $\sigma\mathrm{lcf}\beta > \omega$. Let $f,g:\beta \xrightarrow{1-1} \beta$ be given so that f,g are β-recursive and have β-recursive range. We construct some β-recursive $h:\beta \xrightarrow[\text{onto}]{1-1} \beta$ so that

$$h(x) = y \Longrightarrow (f(x)) = y \vee g(y) = x),$$

which obviously implies

$$(A \equiv^r_1 B \text{ via } f,g \Longrightarrow h[A] = B),$$

for all sets $A,B \subseteq \beta$. We define three subsets of β:

$$x \in X_{even} :\Longleftrightarrow (\exists n \in \omega)(\exists x' \in \beta)((gf)^n(x') = x \text{ and } x' \notin ran(g))$$

$$x \in X_{odd} :\Longleftrightarrow (\exists n \in \omega)(\exists y \in \beta)((gf)^n g(y) = x \text{ and } y \notin ran(f))$$

$$x \in X_{inf} :\Longleftrightarrow \exists \text{ sequence } (s_0, s_1, \ldots) \text{ so that } s_0 = x \text{ and}$$
$$\text{and } (\forall i \in \omega)(f(s_{2i+2}) = s_{2i+1} \text{ and } g(s_{2i+1}) = s_{2i}).$$

It is easily seen that X_{even}, X_{odd}, X_{inf} are a β-recursive partition of β, and that the β-recursive function h defined as follows is total and onto:

$$h(x) = y :\Longleftrightarrow (x \in X_{even} \cup X_{inf} \text{ and } f(x) = y) \vee (x \in X_{odd} \text{ and } g(y) = x).$$

We now turn to the structure of the minimum and the maximum m-degree of β-r.e. sets. The situation in the set of the β-recursive sets is nearly the same as in CRT (cf. e.g., Odifreddi [15]).

4.2. Theorem.

(i) The structure of the β-recursive 1-r-degrees under the $\leq_1^r$-ordering is as follows:

$$\langle 0\rangle < \langle 1\rangle < \ldots < \langle\omega\rangle < \ldots < \langle\delta\rangle < \ldots < \langle\{2x \mid x<\beta\}\rangle$$

$$\langle L_\beta\rangle < \langle L_\beta - 1\rangle < \ldots < \langle L_\beta - \omega\rangle < \ldots < \langle L_\beta - \delta\rangle < \ldots < \langle\{2x \mid x<\beta\}\rangle$$

($\delta < \beta$ is a β-cardinal).

We know by 4.1. that the 1-r-degree of a set A coincides with its isomorphism type $\langle A\rangle$.)

(ii) If $\beta^* \geq \sigma 1\mathrm{cf}\beta$, then the β-recursive 1-degrees and the β-recursive isomorphism types coincide. The $\leq_1$-ordering of these degrees is the same as in (i).

(iii) If $\sigma 1\mathrm{cf}\beta > \beta^*$, then the isomorphism types $\langle\beta^*\rangle$, $\langle\{2x \mid x<\beta\}\rangle$, and $\langle\beta - \beta^*\rangle$ are all contained in the same 1-degree. All the other β-recursive isomorphism types are 1-degrees.

(The proof can be found in [4].)

We now turn to the study of the maximum β-r.e. m-degree. ($A \leq_m B$ if $f^{-1}[B] = A$ for some β-rec. $f{:}\beta \longrightarrow \beta$. A is m-complete if A is β-r.e. and for all β-r.e. sets $B \subseteq \beta$ holds $B \leq_m A$.) The situation is entirely similar to that in CRT if $\beta^* = \sigma 1\mathrm{cf}\beta$. Here the m-complete sets form a single isomorphism type. If $\beta^* < \sigma 1\mathrm{cf}\beta$, there are 1-complete sets which are not β-recursively isomorphic. But at least the m-complete sets are the same as the 1-complete sets.

To prove these facts, we use as an aid the notion of a "creative set" or "constructively non-β-recursive set." These sets play a similar role in the β-r.e. m-degrees as they do in the m-degrees in CRT. We prove that, for all β, the creative sets are the same as the m-complete sets.

Since the definition of the notion "creative set" requires some notion of an "acceptable numbering" of the partial β-recursive functions and the β-r.e. sets, we first study some aspects of such numberings and prove some elementary facts, e.g., the recursion theorem. The main result concerning numberings is that they are all equivalent in a strong sense. The major problem with these notions is to find the correct definitions. Nearly all proofs of the propositions below are adaptations of methods from CRT. (Complete proofs may be found in [4].)

4.3. Definition. (Acceptable numberings)
A two-placed partial function g with $\mathrm{dom}(g) \subseteq \beta^* \times L_\beta$ is called an acceptable numbering if and only if

(1) g is partial β-recursive
(2) for all partial β-recursive functions h with $\mathrm{dom}(h) \subseteq \beta^* \times L_\beta$ there is a β-recursive function $r:\beta^* \xrightarrow{1-1} \beta^*$ with $\mathrm{ran}(r)$ β-recursive such that

$$h(e,x) \simeq g(r(e),x) \quad \text{for all} \quad e < \beta^* \quad \text{and} \quad x \in L_\beta.$$

Remark. (Existence and Uniqueness)
There exists an acceptable numbering. (Such a numbering can be constructed from a universal β-recursive function, as may be found in Devlin [3].)

Any two acceptable numberings are β-recursively isomorphic in the following sense:

If g and h are acceptable numberings, then there is some β-recursive (total) function $t:\beta^* \xrightarrow[\text{onto}]{1-1} \beta^*$ such that

$$g(e,x) \simeq h(t(e),x) \quad \text{for all} \quad e < \beta^* \quad \text{and all} \quad x \in L_\beta.$$

(t can be constructed as in the proofs of Theorems 3.1 and 4.1.)

4.4. Proposition.
Let g be an acceptable numbering. Then g has the following properties:

(1) (The enumeration property)

If f is any partial β-recursive function, then for some $e < \beta^*$ holds:

$$f(x) \simeq g(e,x) \quad \text{for all} \quad x \in L_\beta.$$

(Any such e is called an index for f with respect to g.)

(2) (The iteration property)

There is a β-recursive function $s:\beta^* \times L_\beta \longrightarrow \beta^*$ such that for all $e < \beta^*$ and all $z,x \in L_\beta$ holds

$$g(e,(z,x)) \simeq g(s(e,z),x).$$

s can be assumed to be one-one.

If $\sigma lcf\beta \leq \beta^*$, we can even find such an s with β-recursive range.

(3) (The recursion theorem - with parameter)

If f is a partial β-recursive function with $dom(f) \subseteq \beta^* \times L_\beta \times L_\beta$, then there is a β-recursive $n:L_\beta \longrightarrow \beta^*$ such that

$$f(n(a),a,x) \simeq g(n(a),x), \text{ for all } a,x \in L_\beta.$$

n can be assumed to be one-one.

In particular, if f is a partial β-recursive function with $dom(f) \subseteq \beta^* \times L_\beta$, then for some $e < \beta^*$ holds $f(e,x) \simeq g(e,x)$ for all $x \in L_\beta$.

Proof. Immediate from the definition.

4.5. Lemma.
Let $\sigma lcf\beta \geq \beta^*$. Then the definition of an acceptable numbering can be weakened as follows:

Assume that g is a partial β-recursive function, and that for g the following condition is satisfied:

> For all partial β-recursive h with $dom(h) \subseteq \beta^* \times L_\beta$ there is some β-recursive function $r:\beta^* \longrightarrow \beta^*$ such that $h(e,x) \simeq g(r(e),x)$, for all $e < \beta^*$, all $x \in L_\beta$.

Then g is an acceptable numbering. (This is proved as in CRT, using the recursion theorem 4.4(3); cf. Schnorr [18].) In order to be able to use the familiar notation for the enumerations of the partial β-recursive functions and the β-r.e. sets, we single out one acceptable numbering and use it as our standard numbering. In view of the remark following Definition 4.3, it does not matter which we choose.

4.6. Definition.
Let g be some fixed acceptable numbering.

(1) For each $e < \beta^*$ let $\{e\}$ be the partial function defined by

$$\{e\}(x) :\simeq g(e,x), \text{ for all } x \in L_\beta.$$

We can think of $\{e\}$ as an n-placed function as well:

(2) $\{e\}(x_1,\ldots,x_n) :\simeq g(e,(x_1,\ldots,x_n))$ for all $n \geq 2$, all $x_1,\ldots,x_n \in L_\beta$.

<u>4.7. Proposition.</u> (The s-m-n-theorem)
For all $m,n > 0$ there is a β-recursive function

$$S_n^m : \beta^* \times L_\beta^m \xrightarrow{1-1} \quad \text{such that}$$

$$\{e\}(y_1,\ldots,y_m,x_1,\ldots,x_n) \simeq \{S_n^m(e,y_1,\ldots,y_m)\}(x_1,\ldots,x_n)$$

for all $e < \beta^*$, all $y_j, x_i \in L_\beta$. If $\beta^* \geq \sigma lcf\beta$, S_n^m can be assumed to have β-recursive range.

(This follows from the iteration property 4.4(2).)

<u>4.8. Remark.</u>
The notion of an acceptable numbering as it is defined in 4.3 (cf. Schnorr [18]) is essentially the same as that used in Rogers [16]:

A partial function g with $dom(g) \subseteq \beta^* \times L_\beta$ is an acceptable numbering if and only if there are β-recursive functions $r,s: \beta^* \longrightarrow \beta^*$ such that

(1) $g(e,x) \simeq \{r(e)\}(x)$ for all $e < \beta^*$, all $x \in L_\beta$.

(2) $\{e\}(x) \simeq g(s(e),x)$ for all $e < \beta^*$, all $x \in L_\beta$.

(3) s is one-one and has β-recursive range.

If $\sigma lcf\beta \geq \beta^*$, this equivalence holds as well if we drop (3). (The proof uses 4.5.)

<u>4.9. Definition.</u> (Enumeration of the β-r.e. sets; creative sets)

(1) $W_e := dom(\{e\}) = \{x \in L_\beta | \{e\}(x) \text{ is defined}\}$, for $e < \beta^*$.

(2) $K := \{e < \beta^* | e \in W_e\}$.

(3) A β-r.e. set $A \subseteq L_\beta$ is called creative if and only if there is a partial β-recursive function f with $dom(f) \subseteq \beta^*$ such that

$$(\forall e < \beta^*)(W_e \cap A = \emptyset \longrightarrow e \in dom(f) \quad \text{and} \quad f(e) \notin W_e \cup A).$$

We say then that A is creative via f.

<u>4.10. Proposition.</u>

(1) A set $B \subseteq L_\beta$ is β-r.e. if and only if $W_e = B$ for some $e < \beta^*$. (We say that e is a β-r.e. index for B if $W_e = B$.)

(2) Creative sets are not β-recursive.

(3) K is creative.

(4) The notion of a "creative set" does not depend on the particular numbering we have chosen.

(Proofs are as in CRT. (4) follows from the remark following Definition 4.3.)

4.11. Lemma.

(1) If $A, B \subseteq L_\beta$ are β-r.e., A is creative, and $A \leq_m B$, then B is creative.

(2) All m-complete sets are creative.

(Proofs as in CRT.)

4.12. Lemma.

A β-r.e. set $A \subseteq L_\beta$ is creative if and only if it is creative via some f with $\mathrm{dom}(f) = \beta^*$.

(Proof as in CRT, using the recursion theorem 4.4.(3).)

4.13. Corollary.

(1) A β-r.e. set $A \subseteq L_\beta$ is creative if and only if it is m-complete.

(2) A β-r.e. set $A \subseteq L_\beta$ is 1-r-complete if and only if it is a creative cylinder.

(The proof uses the recursion theorem 4.4.(3), and 4.11., 4.12.)
(A set $A \subseteq L_\beta$ is called a cylinder if $A = h[\{\langle x,y\rangle : x \in B,\ y \in L_\beta\}]$ for some β-recursive permutation h of L_β, and some β-recursive pairing functions $\langle\ ,\ \rangle : L_\beta \times L_\beta \xrightarrow[\text{onto}]{1\text{-}1} L_\beta$, and some $B \subseteq L_\beta$. For this notion, cf. Rogers [16].)

We can improve the result 4.13. in the case that β is not strongly inadmissible, i.e., if $\beta^* \leq \sigma\mathrm{lcf}\beta$:

4.14. Lemma.

Let $\beta^* \leq \sigma\mathrm{lcf}\beta$.
A β-r.e. set $A \subseteq L_\beta$ is creative if and only if it is creative via some β-recursive function f which has domain β^* and is one-one.

(The proof combines the method of proving the corresponding theorem of CRT with manipulations of β-r.e. indices of sets involving the recursion theorem, similar to those used for handling indices of hyperarithmetic sets, cf. Hinman [10].)

4.15. Theorem.

Let $\beta^* \leq \sigma\mathrm{lcf}\beta$. Then for all $A \subseteq L_\beta$ holds A is creative if and only if A is 1-complete if and only if A is m-complete.

(Proof as in CRT, uses 4.13. and 4.14.)

The following two results deal with special kinds of not strongly inadmissible β. If $\sigma\mathrm{lcf}\beta = \beta^*$, the situation inside the maximum β-r.e. m-degree is the same as in CRT:

4.16. Theorem.

Let $\sigma\mathrm{lcf}\beta = \beta^*$.
Then all creative sets are β-recursively isomorphic.

(Proof as in CRT, involving 4.7. and 4.10.)

Theorem 4.17. tells us that in the case $\beta^* < \sigma\mathrm{lcf}\beta$ all creative sets which are contained in some i-finite set are β-recursively isomorphic (cf. Kripke [12]). The sets which are creative and cylinders form a different isomorphism type (of 1-complete sets).

4.17. Theorem.

Let $\beta^* < \sigma\mathrm{lcf}\beta$. Then the following assertions hold:

(1) If $A,B \subseteq L_\beta$ are creative sets and $a,b \in L_\beta$ are i-finite such that $A \subseteq a$ and $B \subseteq b$, then A and B are β-recursively isomorphic. (K is such a set.)

(2) If $A,B \subseteq L_\beta$ are creative cylinders, then A and B are β-recursively isomorphic. ($K \times L_\beta$ is a creative cylinder.)

(3) A β-r.e. set A is creative if and only if for some function $f \in L_\beta$ holds:

$$\mathrm{dom}(f) = \beta^* \text{ and } f:\beta^* \xrightarrow{1-1} L_\beta \text{ and } f[K] = A \cap \mathrm{ran}(f).$$

(4) A set $A \subseteq L_\beta$ is creative if and only if there is some β-r.e. set $B \subseteq L_\beta - \beta^*$ such that $A \equiv K \cup B$.

(The proof employs ideas from 4.1., and uses 4.11., 4.13., 4.15.)

REFERENCES

[1] J. K. Barwise, Admissible sets and structures (Springer, Berlin, 1975).

[2] K. J. Devlin, Aspects of constructibility, Lecture Notes in Mathematics 354 (Springer, Berlin, 1973).

[3] K. J. Devlin, Constructibility, in: J. K. Barwise (ed.), Handbook of Mathematical Logic (North-Holland, Amsterdam, 1977).

[4] M. Dietzfelbinger, Strong reducibilities in α- and β-recursion theory (Diplomarbeit, München, 1982).

[5] F. R. Drake, Set theory: An introduction to large cardinals (North-Holland, Amsterdam, 1974).

[6] J. E. Fenstad, General recursion theory: an axiomatic approach (Springer, Berlin, 1980).

[7] S. D. Friedman, Recursion on inadmissible ordinals, Ph.D. Thesis, M.I.T., Cambridge, MA., 1976.

[8] S. D. Friedman, β-recursion theory, Trans. Am. Math. Soc. 255 (1979), 173-200.

[9] S. D. Friedman and G. E. Sacks, Inadmissible recursion theory, Bull. Am. Math. Soc., 83 (1977), 255-256.

[10] P. G. Hinman, Recursion-theoretic hierarchies (Springer, Berlin, 1978).

[11] R. B. Jensen, The fine structure of the constructible hierarchy, Ann. Math. Logic, 4 (1972), 229-308.

[12] S. Kripke, Transfinite recursions on admissible ordinals II (abstract), J. Symb. Logic, 29 (1964), 161-162.

[13] W. Maass, Inadmissibility, tame r.e. sets and the admissible collapse, Ann. Math. Logic, 13 (1978), 149-170.

[14] W. Maass, Recursively invariant β-recursion theory, Ann. Math. Logic, 21 (1981), 27-73.

[15] P. Odifreddi, Strong reducibilities, Bull. (New Series) Am. Math. Soc., 4 (1981), 37-86.

[16] H. Rogers, Jr., Theory of recursive functions and effective computability (McGraw-Hill, New York, 1977).

[17] G. E. Sacks and S. G. Simpson, The α-finite injury method, Ann. Math. Logic, 4 (1972), 323-367.

[18] C. P. Schnorr, Rekursive Funktionen und ihre Komplexität (Teubner, Stuttgart, 1974).

[19] R. A. Shore, Σ_n sets which are Δ_n-incomparable (uniformly), J. Symb. Logic, 39 (1974), 295-304.

[20] R. A. Shore, α-recursion theory, in: J. K. Barwise (ed.), Handbook of Mathematical Logic (North-Holland, Amsterdam, 1974).

[21] R. A. Shore, Splitting an α-r.e. set, Trans. Am. Math. Soc., 204 (1975), 65-78.

[22] S. G. Simpson, Degree theory on admissible ordinals, in: J. E. Fenstad, P. G. Hinman (eds.), Generalized recursion theory (North-Holland, Amsterdam, 1974).

[23] S. G. Simpson, Short course on admissible recursion theory, in: J. E. Fenstad, G. E. Sacks (eds.), Generalized recursion theory II (North-Holland, Amsterdam, 1978).