

## VARIATIONS ON PROMPTLY SIMPLE SETS

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**§1. Introduction.** In this paper we answer the question of whether all low sets with the splitting property are promptly simple. Further we try to make the role of lowness properties and prompt simplicity in the construction of automorphisms of the lattice of r.e. (recursively enumerable) sets more perspicuous. It turns out that two new properties of r.e. sets, which are dual to each other, are essential in this context: the prompt and the low shrinking property.

In an earlier paper [4] we had shown (using Soare's automorphism construction [10] and [12]) that all r.e. generic sets are automorphic in the lattice  $\mathcal{E}$  of r.e. sets under inclusion. We called a set  $A$  *promptly simple* if  $\bar{A}$  is infinite and there is a recursive enumeration of  $A$  and the r.e. sets  $(W_e)_{e \in \mathbb{N}}$  such that if  $W_e$  is infinite then there is some element (or equivalently: infinitely many elements)  $x$  of  $W_e$  such that  $x$  gets into  $A$  "promptly" after its appearance in  $W_e$  (i.e. for some fixed total recursive function  $f$  we have  $x \in A_{f(s)}$ , where  $s$  is the stage at which  $x$  entered  $W_e$ ). Prompt simplicity in combination with lowness turned out to capture those properties of r.e. generic sets that were used in the mentioned automorphism result. In a following paper with Shore and Stob [7] we studied an  $\mathcal{E}$ -definable consequence of prompt simplicity: the splitting property. One says that  $A$  has the *splitting property* if every r.e. set  $B$  can be split into r.e. sets  $B_0, B_1$  with  $B_0 \subseteq A$  such that  $B_0, B_1$  is a "Friedberg splitting" of  $B$  (i.e.  $B = B_0 \cup B_1, B_0 \cap B_1 = \emptyset$ , and if  $W$  is r.e. and  $W - B$  is not r.e. then  $W - B_0$  and  $W - B_1$  are also not r.e.). The class of r.e. sets that are not hyperhypersimple but have the splitting property became the first example of an  $\mathcal{E}$ -definable class of r.e. sets whose degrees split the high degrees. More recently Ambos-Spies, Jockusch, Shore and Soare [1] proved that in fact the degrees of nonhyperhypersimple sets that have the splitting property coincide with the degrees of promptly simple sets. In particular, sets of low degree are not hyperhypersimple. Since sets of low degree have received particular attention over the last few years, it was natural to ask whether at least for sets of low degree prompt simplicity and the

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splitting property coincide. This question was first studied by Stob and Soare, who noticed that standard arguments do not suffice for an answer.

We show in this paper that there are in fact low sets with the splitting property that are not promptly simple (Corollary 3.6). We have decided to present this result in a way that yields some pleasant side effects. We define in §2 the prompt shrinking property and show in Lemma 2.4 that this property is implied by prompt simplicity. In Theorem 2.7 we construct a set of low degree which has the prompt shrinking property but is not promptly simple. The proof of Theorem 2.7 uses an interesting technique from the recent monstrous injury constructions (Lachlan [3], Harrington [2]). A system of many different strategies is played against the opponent, where the failure of one strategy makes it easier for another strategy in this system to overcome the opponent. In §3 we show that any two low sets with the prompt shrinking property are automorphic in  $\mathcal{E}$ . This generalizes the above-mentioned automorphism result for promptly simple low sets and shows for the first time that there are sets in this orbit that are not promptly simple (Corollary 3.5). This appears to be of some interest since in the meantime promptly simple sets have proven to be useful independent of their automorphism properties (see e.g. [1]). We believe that the new automorphism result (Corollary 3.3) stretches the available techniques to their limit. If there are in addition sets without the prompt shrinking property in this orbit, a proof of this fact is likely to provide novel insights.

Along the way we also analyze more closely the role of lowness properties in automorphism constructions. By Soare [12] it is sufficient for the automorphism result that the considered promptly simple sets are semi-low ( $A$  is *semi-low* if  $\{e \mid W_e \cap \bar{A} \neq \emptyset\} \leq_T 0'$ ). We introduce in §2 the low shrinking property, which is completely dual to the previously mentioned prompt shrinking property. This appears to be noteworthy because so far lowness properties seemed to have no relationship to prompt simplicity. We show in Lemma 2.5 that semi-low implies the low shrinking property. According to Remark 2.6 the low shrinking property implies semi-low<sub>1.5</sub> ( $A$  is semi-low<sub>1.5</sub> if  $\{e \mid W_e \cap \bar{A} \text{ finite}\} \leq_1 \emptyset''$ ). The use of the low shrinking property simplifies the previously existing automorphism construction for promptly simple semi-low sets [4], and also gives rise to new applications. For the new automorphism construction we consider any two r.e. sets  $A, B$  for which an isomorphism between their lattices of supersets exists that satisfies a very weak covering property ((\*) of Lemma 3.1; note that this condition only talks about elements of  $\bar{A}$  and  $\bar{B}$ ). The “shrinking lemma” (Lemma 3.1) tells us that if  $A, B$  have in addition the low and the prompt shrinking property, then we can shrink the image sets of the given isomorphism of the lattices of supersets in such a way that it satisfies in addition property (\*\*) of Lemma 3.1, which is Soare’s famous covering property [10]. Once this is achieved we can apply Soare’s extension theorem [10] and extend the isomorphism between supersets of  $A$  and  $B$  to an automorphism of  $\mathcal{E}$  that maps  $A$  on  $B$ . An obvious advantage of this new procedure is the fact that it is substantially more easy to construct isomorphisms that have the very weak covering property (\*) instead of the weak covering property from [4]. Such isomorphisms are e.g. constructed in [5] for any two sets  $A, B$  that are semi-low<sub>1.5</sub>. Previously one had to construct for sets  $A, B$  that are semi-low a more special isomorphism that satisfies the somewhat stronger “weak covering property” from [4]; see Soare [12].

Essentially one had to do some of the work of the shrinking lemma within the isomorphism construction (which is already complex enough on its own).

In addition, the new arrangement of the automorphism construction is more flexible. In [6] we describe a variation of it that yields the first nontrivial automorphism of  $\mathcal{M}$  ( $\mathcal{M}$  is the lattice of an interval of  $\mathcal{E}$  that is bounded by a major subset; see [8]). In  $\mathcal{M}$  we have no counterpart to Soare's isomorphism construction for semi-low sets, only from [8] a counterpart to the weaker construction for semi-low<sub>1.5</sub> sets that achieves the very weak covering property. Therefore in  $\mathcal{M}$  we have to use the way via the shrinking lemma and the low shrinking property.

Concerning notation we write  $A =^* B$  if the sets  $A, B$  agree except for finitely many elements. We write  $A^*$  for the equivalence class of  $A$  with respect to the equivalence relation  $=^*$ . We consider

$$\mathcal{E}(S) = \{W \cap S \mid W \text{ r.e.}\},$$

which forms a lattice under set-theoretic union and intersection. Instead of  $\mathcal{E}(N)$  we write  $\mathcal{E}$ .  $\mathcal{E}^*(S)$  and  $\mathcal{E}^*$  are the corresponding quotient lattices modulo the ideal of finite sets.

We use the convention that a set is meant to be r.e. unless we say otherwise. Further, deviating from the original definition in Soare [11], we say that an r.e. set (instead of its complement) is semi-low, semi-low<sub>1.5</sub>, etc.

As usual,  $U \setminus V$  is the set of elements which are enumerated in  $U$  while they are not (yet) in  $V$ ;  $U \searrow V := (U \setminus V) \cap V$ .

In a simultaneous enumeration of an array of r.e. sets we assume that at every stage at most one element is enumerated in at most one set.

## §2. Shrinking properties and prompt simplicity.

DEFINITION 2.1. We say that  $A$  has the *prompt shrinking property* if for any simultaneous enumeration of r.e. sets  $(X_i)_{i \in N}$  we can effectively assign to every  $X_i$  an r.e. set  $X_i^p \subseteq X_i$  with  $X_i^p \cap \bar{A} =^* X_i \cap \bar{A}$  such that for every  $j \in N$  and every finite  $K \subseteq N$

$$\left( X_j \setminus \left( \bigcup_{i \in K} X_i \right) \right) \cap \bar{A} \text{ infinite} \Rightarrow \left( X_j - \left( \bigcup_{i \in K} X_i^p \right) \right) \cap A \text{ infinite.}$$

DEFINITION 2.2. We say that  $A$  has the *low shrinking property* if for any simultaneous enumeration of r.e. sets  $(X_i)_{i \in N}$  we can effectively assign to every  $X_i$  an r.e. set  $X_i^l \subseteq X_i$  with  $X_i^l \cap \bar{A} =^* X_i \cap \bar{A}$  such that for every  $j \in N$  and every finite  $K \subseteq N$

$$\left( \left( \bigcap_{i \in K} X_i^l \right) - X_j \right) \cap A \text{ infinite} \Rightarrow \left( \left( \bigcap_{i \in K} X_i \right) \cap \bar{A} \right) \text{ infinite.}$$

REMARK 2.3. One sees easily that both the prompt and the low shrinking property are recursively invariant in the sense of H. Rogers, Jr. [9].

LEMMA 2.4. Assume  $A$  is promptly simple. Then  $A$  has the prompt shrinking property.

PROOF. Since  $A$  is promptly simple there exists (see [4]) a recursive function  $g$  such that for every  $i \in N$

$W_{(g(i))_0}, W_{(g(i))_1}$  is a splitting of  $W_i$ ,  
 $W_{(g(i))_0} \subseteq A$  and  
 $W_i$  infinite  $\Rightarrow W_{(g(i))_0}$  infinite.

Assume that a simultaneous enumeration of r.e. sets  $(X_i)_{i \in N}$  is given. Fix a recursive function  $I$  such that for every  $j \in N$  and every finite set  $K \subseteq N$

$$W_{I(j,K)} = \left( X_j \setminus \left( \bigcup_{k \in K} X_k \right) \right).$$

For every  $i \in N$  we enumerate an r.e. set  $X_i^P \subseteq X_i$  as follows. We enumerate  $x \in X_i$  into  $X_i^P$  iff  $x \in W_{(g(I(j,K)))_1}$  for all  $\langle j, K \rangle \leq x$  with  $i \in K$  and  $x \in (X_j \setminus (\bigcup_{k \in K} X_k))$ . Note that after  $x$  appears in  $X_i$  we can check immediately for which  $\langle j, K \rangle \leq x$  with  $i \in K$  we have  $x \in (X_j \setminus (\bigcup_{k \in K} X_k))$ . Since  $W_{(g(I(j,K)))_0} \subseteq A$  for every  $j, k$ , we have  $X_i \cap \bar{A} = X_i^P \cap \bar{A}$ .

Consider some  $j, K$  with  $(X_j \setminus (\bigcup_{k \in K} X_k)) \cap \bar{A}$  infinite. Assume that  $X_j \setminus (\bigcup_{k \in K} X_k)$  is infinite (otherwise  $X_j - (\bigcup_{k \in K} X_k)$  is an infinite r.e. set and thus has obviously an infinite intersection with the simple set  $A$ ). Then  $W_{(g(I(j,K)))_0} \subseteq A$  is infinite as well. Take any  $x \in W_{(g(I(j,K)))_0}$  with  $x \geq \langle j, K \rangle$ . According to the definition of  $X_i^P$  this  $x$  is not enumerated in any  $X_i^P$  with  $i \in K$ . Thus  $(X_j - (\bigcup_{k \in K} X_k^P)) \cap A$  contains almost all elements of the infinite set  $W_{(g(I(j,K)))_0}$ .

LEMMA 2.5. Assume  $A$  is semi-low. Then  $A$  has the low shrinking property.

PROOF. Assume that a simultaneous enumeration of sets  $(X_i)_{i \in N}$  is given. Fix a recursive function  $I$  such that for every  $j \in N$  and every finite set  $K \subseteq N$

$$W_{I(j,K)} = \left( \bigcap_{k \in K} X_k \right) \setminus X_j.$$

Since  $A$  is semi-low, we can speed up the enumeration of  $A$  so that we have for the induced simultaneous enumeration of  $(W_{I(j,K)})_{j \in N, K \text{ finite}}$ :

$$W_{I(j,K)} \setminus A \text{ infinite} \Leftrightarrow W_{I(j,K)} - A \text{ infinite.}$$

For every  $i \in N$  we enumerate an r.e. set  $X_i^L \subseteq X_i$  as follows. We enumerate  $x \in X_{i,s} - X_{i,s-1}$  into  $X_i^L$  iff  $x \in W_{I(j,K)} \setminus A$  for all  $\langle j, K \rangle \leq x$  with  $i \in K$  and  $x \in (\bigcap_{k \in K} X_{k,s}) - X_{j,s}$ . It is obvious from this definition that  $X_i^L \cap \bar{A} = X_i \cap \bar{A}$ .

Consider some  $j, K$  with  $((\bigcap_{k \in K} X_k^L) - X_j) \cap A$  infinite. Fix some  $i \in K$  such that

$$S = \left( \left( \bigcap_{k \in K} X_k^L \right) - X_j \right) \cap A \cap \left( \left( \bigcap_{\substack{k \in K \\ k \neq i}} X_k \right) \setminus X_i \right)$$

is infinite. Every element  $x$  of  $S$  is in  $X_i^L$  and if  $x \geq \langle j, K \rangle$  we thus have  $x \in W_{I(j,K)} \setminus A$ . Therefore  $W_{I(j,K)} \cap \bar{A}$  is infinite by the choice of the enumeration of  $A$ .

REMARKS 2.6. 1) If  $A$  has the prompt shrinking property and  $\bar{A}$  is infinite, then  $A$  is obviously  $d$ -simple (although perhaps not uniformly  $d$ -simple).

2) There is no apparent way to weaken the prompt shrinking property in such a way that one can still prove Corollary 3.3. For example if we replace

$$\left( X_j \bigcup_{i \in K} X_i \right) \cap \bar{A} \text{ infinite} \Rightarrow \left( X_j - \bigcup_{i \in K} X_i^P \right) \cap A \text{ infinite}$$

by the weaker demand

$$\left( X_j - \bigcup_{i \in K} X_i \right) \cap \bar{A} \text{ infinite} \Rightarrow \left( X_j - \bigcup_{i \in K} X_i^P \right) \cap A \text{ infinite}$$

then we arrive at a property that is implied by uniform  $d$ -simplicity and thus not sufficient for Corollary 3.3 (this follows from the proof of Theorem 2.8. in [7]).

3) If  $A$  has the low shrinking property then  $A$  is semi-low $_{1,5}$ : Apply the low shrinking property of  $A$  to the array  $(W_e)_{e \in N}$ ; then

$$W_e^L \text{ infinite} \Leftrightarrow W_e \cap \bar{A} \text{ infinite.}$$

**THEOREM 2.7.** *There is an r.e. set  $A$  of low degree such that  $\bar{A}$  is infinite,  $A$  has the prompt shrinking property but  $A$  is not promptly simple.*

**PROOF.** There is a simultaneous enumeration of all simultaneously enumerable arrays  $(X_i)_{i \in N}$  which preserves the order of enumeration of each array  $(X_i)_{i \in N}$ . Thus in order to give  $A$  the prompt shrinking property, we only have to consider one simultaneous enumeration of a certain array  $(X_i)_{i \in N}$ .

In order to make the degree of  $A$  low we consider the usual restraint function  $r$ :

$$r(e, s) := \max \{ \text{use}(\{i\}_s^{A_s(i)}) \mid i \leq e \text{ and } \{i\}_s^{A_s(i)} \downarrow \}$$

where  $A_s$  is the set of elements in  $A$  at the end of stage  $s$ . We make sure that for every  $e \in N$  there are only finitely many  $x$  and  $s$  such that  $x \leq r(e, s)$  and  $x \in A_{s+1} - A_s$ .

In order to give  $A$  the prompt shrinking property we construct for every  $e \in N$  an r.e. set  $X_e^P \subseteq X_e$  with  $X_e^P \cap \bar{A} =^* X_e \cap \bar{A}$  such that all the following positive requirements  $P_{i,K,n}$  (for  $i, n \in N$  and  $K \subseteq N$  finite) are satisfied:

$$P_{i,K,n}: X_i \searrow \bigcup_{k \in K} X_k \text{ infinite} \Rightarrow \left| \left( X_i - \left( \bigcup_{k \in K} X_k^P \right) \right) \cap A \right| > n.$$

During the construction we try to satisfy  $P_{i,K,n}$  as follows. As soon as we see an element  $x$  of  $X_i \searrow \bigcup_{k \in K} X_k$  we try to keep it out of  $X_k^P$  for all  $k \in K$  and to push it into  $A$ . We collect all elements of  $X_k$  that we want to keep out of  $X_k^P$  and which we want to push into  $A$  in a separate r.e. set  $X_k^A$  (the superscript  $A$  indicates that its elements are targeted for  $A$ ). Often we do not succeed in pushing an element of  $X_k^A$  into  $A$  because we do not get it past some negative restraint. In this situation we are forced to enumerate almost always the respective element of  $X_k^A$  in addition into  $X_k^P$ , since we have to make  $X_k^P \cap \bar{A} =^* X_k \cap \bar{A}$ . Thus we give up trying to use this element to satisfy  $P_{i,K,n}$ .

In order to make  $A$  not promptly simply we have to satisfy nonfinitary negative requirements  $N_k$ . Basically  $N_k$  enumerates an r.e. set  $T_k$  such that in case  $\{k\}$  is a total recursive function the set  $T_k$  is infinite and for almost all  $x, s \in N$  such that  $x$  enters  $T_k$  at stage  $s$  we have  $x \notin A_{\{k\}(s)}$ . Thus  $N_k$  forces us either to restrain almost all elements of  $T_k$  temporarily from  $A$  (until stage  $\{k\}(s) + 1$ ) or to restrain one element of  $T_k$  permanently from  $A$  (in case that  $\{k\}(s) \uparrow$ ).

Since we have to enumerate almost all elements of  $X_k^A - X_k^P$  into  $A$ , we only allow requirements  $N_i$  with  $i \leq k$  to prevent this. Thus a requirement  $N_j$  with  $j > k$  is in general infinitely often injured because all the elements of  $T_j$  may be pushed prematurely into  $A$  for the sake of  $X_k^A$ . Therefore we create instead of one

requirement  $N_j$  for every  $\tau \subseteq j$  a different version  $N_{j,\tau}$  of  $N_j$ . Every  $N_{j,\tau}$  tries to enumerate a suitable infinite set  $T_{j,\tau}$ . But  $N_{j,\tau}$  only places elements into  $T_{j,\tau}$  that are already in  $\bigcap_{i \in \tau} X_i^P$ . As before we only let  $N_{i,\tau}$  keep an element  $x$  of  $T_{i,\tau}$  out of  $A$  which ought to get into  $A$  for the sake of  $X_k^A$  (i.e.  $x \in X_k^A - X_k^P$ ) if  $i \leq k$ . Therefore it may happen as before that all elements of  $T_{j,\tau}$  are pushed prematurely into  $A$  because of some  $X_k^A$  with  $k < j$ . But now we know that  $k \notin \tau$ , since only elements of  $\bigcap_{i \in \tau} X_i^P$  are enumerated into  $T_{j,\tau}$ . Further,  $(\bigcap_{i \in \tau} X_i^P) \cap X_k$  is then infinite, and therefore it is easy to make  $\bigcap_{i \in \tilde{\tau}} X_i^P$  infinite for  $\tilde{\tau} := \tau \cup \{k\}$ . This allows requirement  $N_{j,\tilde{\tau}}$  to find infinitely many elements in  $\bigcap_{i \in \tilde{\tau}} X_i^P$  for its set  $T_{j,\tilde{\tau}}$  and these elements can obviously not be bothered anymore by  $X_k^A$ . Thus requirement  $N_{j,\tau}$  may fail. But—in the terminology of the Chicago School of Recursion Theory—on the pile of dead bodies of witnesses for  $N_{j,\tau}$  we can build a safer working ground for requirement  $N_{j,\tilde{\tau}}$ .

In the construction below we will not mention the sets  $T_{j,\tau}$ . Instead we say that  $N_{j,\tau}$  restrains certain elements at some stages. The set  $T_{j,\tau}$  corresponds to the set of all elements that are restrained by  $N_{j,\tau}$  at some stage.

We will show in Lemmas 2.12, 2.14, 2.15 and 2.16 that the constructed set  $A$  has the desired properties.

CONSTRUCTION. Stage  $s + 1$ . To keep the construction effective we only consider numbers  $\leq s + 1$ .

(1) Assume  $x \in X_{e,s+1} - X_{e,s}$ . If  $x \in A_s$  we put  $x$  in  $X_e^A$ . If  $x \notin A_s$  we check whether there is some  $\langle i, K, n \rangle$  such that

$$e \in K, x \in X_i \setminus \bigcup_{k \in K} X_k, x \in X_{i,s} - \bigcup_{k \in K} X_{k,s}^P$$

and

$$\left| \left( X_{i,s} - \bigcup_{k \in K} X_{k,s}^P \right) \cap A_s \right| \leq n.$$

If such  $\langle i, K, n \rangle$  exists we put  $x$  in  $X_e^A$ . We say that  $P_{i,K,n}$  puts  $x$  in  $X_e^A$  for the least such  $\langle i, K, n \rangle$ . Otherwise we put  $x$  in  $X_e^P$ .

(2) If some  $P_{i,K,n}$  has previously put  $y$  in  $X_j^A$ ,  $y$  is not yet in  $A$  or  $X_j^P$  and  $y \leq r(\langle i, K, n \rangle, s)$ , then we place  $y$  in  $X_j^P$  (i.e. we give up the attempt to satisfy  $P_{i,K,n}$  via  $y$  if this would hurt a lowness requirement of higher priority).

(3) Assume there are  $x \notin A_s, l, m \in N$  and  $\sigma \subseteq l$  such that  $|\bigcap_{j \in \sigma} X_j^P| \leq m$ ,

$$\tilde{\sigma} := \{j \in \sigma \mid x \in X_{j,s}^P\} \subseteq \sigma \subseteq \{j \leq l \mid x \in X_{j,s+1}\}$$

and no  $P_{i,K,n}$  with  $\langle i, K, n \rangle \leq \langle l, \sigma, m \rangle$  has put  $x$  in some  $X_j^A$  with  $j \in \sigma - \tilde{\sigma}$ . Then we enumerate  $x$  in  $X_j^P$  for all  $j \in \sigma - \tilde{\sigma}$  (this is done in order to provide requirement  $N_{l,\sigma}$  with enough suitable elements which it can restrain from  $A$ ).

(4) Every requirement  $N_{k,\tau}$  which did not restrain an element at the end of stage  $s$  starts to restrain (from  $A$ ) the least  $y \geq k$  which it never restrained before such that  $y \notin A_s$  and  $y$  is already in  $\bigcap_{j \in \tau} X_j^P$  (if such  $y$  exists).

(5) Assume  $N_{k,\tau}$  started to restrain an element  $y$  at stage  $t < s$  and either  $y \in A_s$  or the computation  $\{k\}_s(t)$  converges with a value  $< s$ . Then  $N_{k,\tau}$  no longer restrains  $y$ .

(6) For every  $j \in N$  we enumerate into  $A$  all elements that are at the moment in  $X_j^A$  but not in  $X_j^P$  and that are at the moment not restrained by some  $N_{k,\tau}$  with  $k \leq j$ .

End of the construction.

LEMMA 2.8.  $X_e^p \cap \bar{A} = {}^* X_e \cap \bar{A}$  for every  $e \in N$ .

PROOF. We have  $X_e - X_e^p = X_e^A - X_e^p$ , and by step (6) of the construction all elements of  $X_e^A - X_e^p$  that are not permanently restrained by some  $N_{k,\tau}$  with  $k \leq j$  get into  $A$ . According to (4) every  $N_{k,\tau}$  restrains at most one element permanently.

LEMMA 2.9. Assume requirement  $P_{i,K,n}$  is satisfied, i.e.

$$X_i \searrow \bigcup_{k \in K} X_k \text{ infinite} \Rightarrow \left| \left( X_i - \bigcup_{k \in K} X_k^p \right) \cap A \right| > n.$$

Then  $P_{i,K,n}$  puts only finitely many elements in sets  $X_e^A$ .

PROOF.  $P_{i,K,n}$  puts only elements of  $X_i \searrow \bigcup_{k \in K} X_k$  in sets  $X_e^A$ . Thus we only have to consider the case where  $X_i \searrow \bigcup_{k \in K} X_k$  is infinite. Since  $P_{i,K,n}$  is satisfied, this implies that

$$\left| \left( X_i - \bigcup_{k \in K} X_k^p \right) \cap A \right| > n.$$

After the stage where the first  $n + 1$  elements of  $(X_i - \bigcup_{k \in K} X_k^p) \cap A$  have arrived in  $X_i$  and  $A$ ,  $P_{i,K,n}$  puts no more elements in any  $X_e^A$ .

LEMMA 2.10. Assume  $P_{i,K,n}$  is satisfied for every  $\langle i, K, n \rangle < e$ . Then there are only finitely many  $x$  and  $s$  such that  $x \in A_{s+1} - A_s$  and  $x \leq \text{use}(\{e\}_s^{A_s}(e))$ . Thus

$$\sup_{s \in N} r(e, s) < \infty.$$

PROOF. Assume  $x \in A_{s+1} - A_s$  and  $x \leq \text{use}(\{e\}_s^{A_s}(e))$ . Then  $x \leq r(e, s)$ . Since  $x$  gets into  $A$  at stage  $s + 1$ , there is some  $j$  such that  $x \in X_{j,s+1}^A - X_{j,s+1}^p$  and  $x$  was put in  $X_j^A$  by some  $P_{i,K,n}$ . Since  $x$  was not placed in  $X_j^p$  during step (2) of stage  $s + 1$ , we have  $\langle i, K, n \rangle < e$ . Thus by our assumption on  $P_{i,K,n}$  and the previous lemma there are only finitely many such  $x, s$ .

The preceding argument implies that every computation  $\{i\}_s^{A_s}(i)$  with  $i \leq e$  is destroyed only finitely often during the construction. Therefore

$$\sup_{s \in N} r(e, s) < \infty.$$

LEMMA 2.11. Every requirement  $P_{i,K,n}$  is satisfied.

PROOF. Assume for a contradiction that  $\langle i, K, n \rangle$  is minimal such that  $P_{i,K,n}$  is not satisfied. Thus  $X_i \searrow \bigcup_{k \in K} X_k$  is infinite and

$$\left| \left( X_i - \bigcup_{k \in K} X_k^p \right) \cap A \right| \leq n.$$

Since  $A \searrow X_k^p = \emptyset$  for every  $k \in N$ , this implies that for all  $s \in N$

$$\left| \left( X_{i,s} - \bigcup_{k \in K} X_{k,s}^p \right) \cap A_s \right| \leq n.$$

We have  $\sup_{s \in N} r(\langle i, K, n \rangle, s) < \infty$  by the minimal choice of  $\langle i, K, n \rangle$  and Lemma 2.10. Further, for every  $\langle l, \sigma, m \rangle < \langle i, K, n \rangle$  only finitely many elements are placed in some  $X_j^p$  by step (3).

According to steps (1), (2) and (3) of the construction the preceding implies that almost all  $x \in X_i \setminus \bigcup_{k \in K} X_k$  are put into  $X_k^A$  by  $P_{i,K,n}$  and remain in  $X_k^A - X_k^P$  for every  $k \in K$  with  $x \in X_k$ . Further, almost all of these  $x$  are placed in  $A$  according to step (6). Thus  $(X_i - \bigcup_{k \in K} X_k^P) \cap A$  is infinite, a contradiction.

LEMMA 2.12. *A is of low degree.*

PROOF. This follows from Lemma 2.10 and Lemma 2.11.

LEMMA 2.13. *If  $k \in N$  and  $\tau \subseteq k$  is maximal with respect to  $\subseteq$  such that  $\bigcap_{i \in \tau} X_i^P$  is infinite, then only finitely many elements are enumerated into  $A$  while they are restrained by  $N_{k,\tau}$ .*

PROOF. Assume the claim does not hold for  $N_{k,\tau}$ . Thus there exists some  $j \in k - \tau$  such that for infinitely many  $x, s \in N$  the element  $x$  is at the end of stage  $s + 1$  in  $(\bigcap_{i \in \tau} X_i^P) \cap (X_j^A - X_j^P)$  and  $x$  is enumerated into  $A$  during step (6) of stage  $s + 1$ . Since every  $P_{i,K,n}$  puts only finitely many elements in sets  $X_e^A$  according to Lemma 2.9 and Lemma 2.11, we see that  $\bigcap_{i \in \tau \cup \{j\}} X_i^P$  is infinite because of the action at step (3) of these stages  $s + 1$ . This contradicts the maximality of  $\tau$ .

LEMMA 2.14.  *$\bar{A}$  is infinite.*

PROOF. Fix some  $k_0 \in N$ . We show that there is some  $y$  in  $N - A$  with  $y \geq k_0$ . Let  $k \geq k_0$  be some index such that  $\text{domain } \{k\} = \emptyset$ . Choose  $\tau \subseteq k$  maximal such that  $\bigcap_{i \in \tau} X_i^P$  is infinite ( $\tau$  exists since  $\bigcap_{i \in \emptyset} X_i^P = N$ ). By the preceding lemma,  $N_{k,\tau}$  starts to restrain some  $y \geq k$  such that  $y$  is not enumerated into  $A$  while it is restrained by  $N_{k,\tau}$ . This implies  $y \in N - A$  by the choice of  $k$ .

LEMMA 2.15. *A is not promptly simple.*

PROOF. Assume  $A$  is promptly simple. Then there is some total recursive function  $\{k\}$  such that for every  $N_{i,\sigma}$  which restrains infinitely many elements during the construction there are infinitely many  $x, s \in N$  so that  $N_{i,\sigma}$  starts to restrain  $x$  at stage  $s$  and  $x \in A_{\{k\}(s)}$ . Choose  $\tau \subseteq k$  maximal such that  $\bigcap_{i \in \tau} X_i^P$  is infinite. Thus, since  $\{k\}$  is total, the requirement  $N_{k,\tau}$  restrains infinitely many elements  $x$  during the construction. By Lemma 2.13 only finitely many of these  $x$  are enumerated into  $A$  while they are restrained by  $N_{k,\tau}$ . Thus according to step (5) we have  $x \notin A_{\{k\}(s_x)}$  for almost all of these  $x$ , where  $s_x$  is the stage at which  $N_{k,\tau}$  started to restrain  $x$ . Contradiction.

LEMMA 2.16. *A has the prompt shrinking property.*

PROOF. We first note that  $A$  is simple. This follows from the satisfaction of all the requirements  $P_{i,K,n}$  and from the fact that we can assume that for every  $i \in N$  with  $X_i$  infinite there is some  $i' \in N$  with  $X_i \setminus X_{i'}$  infinite.

In order to verify the prompt shrinking property of  $A$  we assume that  $(X_i \setminus (\bigcup_{k \in K} X_k)) \cap \bar{A}$  is infinite. If  $X_i \setminus (\bigcup_{k \in K} X_k)$  is finite we are already finished, since then  $X_i - (\bigcup_{k \in K} X_k)$  is an infinite r.e. set and thus has an infinite intersection with the simple set  $A$ . Thus assume  $X_i \setminus (\bigcup_{k \in K} X_k)$  is infinite. Since for every  $n \in N$  the requirement  $P_{i,K,n}$  is satisfied (Lemma 2.11), this implies that  $(X_i - (\bigcup_{k \in K} X_k^P)) \cap A$  is infinite.

This finishes the proof of Theorem 2.7.

**§3. The shrinking lemma and applications.** If one constructs for some sets  $A, B$  an isomorphism  $\psi: \mathcal{E}(\bar{A}) \rightarrow \mathcal{E}(\bar{B})$  one can hardly avoid (see e.g. [5], [8]) to satisfy in addition property (\*) which occurs in the assumption of the shrinking lemma. On the other hand, property (\*\*) in the conclusion of the shrinking lemma is the

property which is needed to continue the isomorphism  $\psi$  to an automorphism  $\Phi$  of  $\mathcal{E}$  with  $\Phi(A) = B$  via Soare's extension theorem [10].

We review now some standard notations for automorphism constructions (see [12] or [5]). We assume that the two considered sets  $A$  and  $B$  are subsets of two different copies of the natural numbers. On the side of  $A$  we look at the arrays  $(U_i)_{i \in \mathbb{N}}$  and  $(\hat{V}_i)_{i \in \mathbb{N}}$  (respectively  $(V'_i)_{i \in \mathbb{N}}$ ), and on the side of  $B$  at the arrays  $(\hat{U}_i)_{i \in \mathbb{N}}$  (respectively  $(U'_i)_{i \in \mathbb{N}}$ ) and  $(V_i)_{i \in \mathbb{N}}$ . The superscripts “ $\wedge$ ” and “ $\vee$ ” indicate that the set serves as image of the set with the same letter on the other side.

A state  $v$  is a triple  $\langle e, \sigma, \tau \rangle$ , where  $e$  is a natural number and  $\sigma, \tau$  are subsets of  $\{0, \dots, e\}$ . One says that a number  $x$  has state  $v = \langle e, \sigma, \tau \rangle$  with respect to  $(U_i)_{i \in \mathbb{N}}, (\hat{V}_i)_{i \in \mathbb{N}}$  if  $\sigma = \{i \leq e \mid x \in U_i\}$  and  $\tau = \{i \leq e \mid x \in \hat{V}_i\}$ . For a fixed simultaneous enumeration of the involved arrays one says that  $x$  has state  $v = \langle e, \sigma, \tau \rangle$  with respect to  $(U_i)_{i \in \mathbb{N}}, (\hat{V}_i)_{i \in \mathbb{N}}$  at stage  $s$  if  $\sigma = \{i \leq e \mid x \in U_{i,s}\}$  and  $\tau = \{i \leq e \mid x \in \hat{V}_{i,s}\}$ . For numbers  $x$  on the other side we say that  $x$  has state  $v = \langle e, \sigma, \tau \rangle$  with respect to  $(\hat{U}_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}$  at stage  $s$  if  $\sigma = \{i \leq e \mid x \in \hat{U}_{i,s}\}$  and  $\tau = \{i \leq e \mid x \in V_{i,s}\}$ .

For states  $v = \langle e, \sigma, \tau \rangle, v' = \langle e, \sigma', \tau' \rangle$  one says that  $v \geq v'$  (“ $v$  covers  $v'$ ”)  $:\Leftrightarrow \sigma \supseteq \sigma' \wedge \tau \supseteq \tau'$ .

LEMMA 3.1 (“SHRINKING LEMMA”). Assume that the sets  $A$  and  $B$  both have the prompt and low shrinking properties. Further assume that there is a simultaneous enumeration of  $A, B$  and r.e. set  $(U_i)_{i \in \mathbb{N}}, (\hat{V}_i)_{i \in \mathbb{N}}, (\hat{U}_i)_{i \in \mathbb{N}}$  and  $(V_i)_{i \in \mathbb{N}}$  such that

(\*) For all states  $v$ , if infinitely many elements of  $\bar{A}$  have state  $v$  with respect to  $(U_i)_{i \in \mathbb{N}}, (\hat{V}_i)_{i \in \mathbb{N}}$  at some point of the enumeration, then there is a state  $v_1 \leq v$  such that infinitely many elements of  $\bar{B}$  have state  $v_1$  with respect to  $(\hat{U}_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}$  at some point of the enumeration; and the symmetrical counterpart.

Then there is a simultaneous enumeration of  $A, B, (U_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}$  and arrays  $(V'_i)_{i \in \mathbb{N}}, (U'_i)_{i \in \mathbb{N}}$  such that  $V'_i \cap \bar{A} = \hat{V}_i \cap \bar{A}$  and  $U'_i \cap \bar{B} = \hat{U}_i \cap \bar{B}$  for all  $i \in \mathbb{N}$ , and

(\*\*) For all states  $v$ , if infinitely many elements enter  $A$  in state  $v$  with respect to  $(U_i)_{i \in \mathbb{N}}, (V'_i)_{i \in \mathbb{N}}$ , then there is a state  $v_1 \leq v$  such that infinitely many elements enter  $B$  in state  $v_1$  with respect to  $(U'_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}$ , and if infinitely many elements enter  $B$  in state  $v$  with respect to  $(U'_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}$  then there is a state  $v_1 \geq v$  such that infinitely many elements enter  $A$  in state  $v_1$  with respect to  $(U_i)_{i \in \mathbb{N}}, (V'_i)_{i \in \mathbb{N}}$ .

PROOF. For finite sets  $H \subseteq \mathbb{N}$  we set  $Y_H := (\bigcap_{i \in H} U_i) \setminus A$ . For the induced enumeration of  $(Y_H)_{H \text{ finite}}, (\hat{V}_i)_{i \in \mathbb{N}}$  we apply the prompt shrinking property of  $A$  (i.e. the union of both arrays  $(Y_H)_{H \text{ finite}}$  and  $(V_i)_{i \in \mathbb{N}}$  plays the role of  $(X_i)_{i \in \mathbb{N}}$  in Definition 2.1; nevertheless we will make no use of the resulting sets  $Y_H^P$ ). This yields sets  $\hat{V}_i^P \subseteq \hat{V}_i$ .

For  $\hat{Y}_H := (\bigcup_{i \in H} U_i) \setminus A$  we then apply the low shrinking property of  $A$  to the enumeration of  $(\hat{Y}_H)_{H \text{ finite}}, (\hat{V}_i^P)_{i \in \mathbb{N}}$  which results from the given enumeration of  $A, (U_i)_{i \in \mathbb{N}}, (\hat{V}_i)_{i \in \mathbb{N}}$  by letting an element in  $\hat{V}_i^P$  only after it appeared in  $\hat{V}_i$ . This yields the sets  $V'_i := \hat{V}_i^{PL} \subseteq \hat{V}_i^P$ . We enumerate these new sets in such a way that an element appears in  $V'_i$  only after it appeared in  $\hat{V}_i^P$ .

Analogously we apply the prompt shrinking property of  $B$  first to the induced enumeration of  $((\bigcap_{i \in H} V_i) \setminus B)_{H \text{ finite}}, (\hat{U}_i)_{i \in \mathbb{N}}$  to yield sets  $\hat{U}_i^P \subseteq \hat{U}_i$ , and then the low shrinking property of  $B$  to the induced enumeration of

$$\left( \left( \bigcup_{i \in H} V_i \right) \setminus B \right)_{H \text{ finite}}, (\hat{U}_i^P)_{i \in \mathbb{N}}$$

to yield the sets  $U'_i := \hat{U}_i^{PL} \subseteq \hat{U}_i^P$ .

We consider now the induced simultaneous enumeration of

$$A, B, (U_i)_{i \in N}, (V_i)_{i \in N}, (U'_i)_{i \in N} \text{ and } (V'_i)_{i \in N}.$$

Assume that infinitely many elements enter  $A$  in state  $v = \langle e, \sigma, \tau \rangle$  with respect to  $(U_i)_{i \in N}, (V_i)_{i \in N}$ . This implies for  $H := \{0, \dots, e\} - \sigma$  that

$$\left( \bigcap_{i \in \tau} \hat{V}_i^{PL} - \left( \left( \bigcup_{i \in H} U_i \right) \setminus A \right) \right) \cap A \text{ is infinite.}$$

Therefore

$$\left( \left( \bigcap_{i \in \tau} \hat{V}_i^P \right) \setminus \left( \left( \bigcup_{i \in H} U_i \right) \setminus A \right) \right) \cap \bar{A} \text{ is infinite}$$

by the choice of  $\hat{V}_i^{PL} \subseteq \hat{V}_i^P$ . Since every element is already in  $\hat{V}_i$  when it appears in  $\hat{V}_i^P$ , this implies that for some state  $v_1 \leq v$  infinitely many elements of  $\bar{A}$  are at some point in state  $v_1$  with respect to the given enumeration of  $A, (U_i)_{i \in N}, (\hat{V}_i)_{i \in N}$ .

According to hypothesis (\*) of the lemma there then exists some state  $v_2 = \langle e, \sigma_2, \tau_2 \rangle \leq v_1$  such that infinitely many elements of  $\bar{B}$  are at some point in state  $v_2$  with respect to  $(\hat{U}_i)_{i \in N}, (V_i)_{i \in N}$ . Thus the set

$$\left( \left( \left( \bigcap_{i \in \tau_2} V_i \right) \setminus B \right) \setminus \left( \bigcup_{i \in \{0, \dots, e\} - \sigma_2} \hat{U}_i \right) \right) \cap \bar{B} \text{ is infinite.}$$

By the choice of  $\hat{U}_i^P \subseteq \hat{U}_i$  this implies that

$$\left( \left( \left( \bigcap_{i \in \tau_2} V_i \right) \setminus B \right) - \bigcup_{i \in \{0, \dots, e\} - \sigma_2} \hat{U}_i^P \right) \cap B \text{ is infinite.}$$

Since  $U'_i = \hat{U}_i^{PL} \subseteq \hat{U}_i^P$ , every element of the preceding infinite set enters  $B$  in some state  $v_3 \leq v_2$  with respect to  $(U'_i)_{i \in N}, (V_i)_{i \in N}$ .

The symmetrical counterpart of the covering property (\*\*) is proved analogously.

**THEOREM 3.2.** *Assume that the sets  $A$  and  $B$  have the prompt and low shrinking properties. Further assume that there is a simultaneous enumeration of  $A, B$  and r.e. sets  $(U_i)_{i \in N}, (\hat{V}_i)_{i \in N}, (\hat{U}_i)_{i \in N}$  and  $(V_i)_{i \in N}$  such that for all  $i \in N$   $U_i =^* W_i$  and  $V_i =^* W_i$ , for every state  $v$  infinitely many elements of  $\bar{A}$  have state  $v$  with respect to  $(U_i)_{i \in N}, (\hat{V}_i)_{i \in N}$  iff infinitely many elements of  $\bar{B}$  have state  $v$  with respect to  $(\hat{U}_i)_{i \in N}, (V_i)_{i \in N}$  and such that property (\*) of the shrinking lemma holds. Then there is an automorphism  $\Phi$  of  $\mathcal{E}$  such that  $\Phi(A) = B$ .*

**PROOF.** Combine the shrinking lemma with Soare's extension theorem (Theorem 2.2 in [10]).

**COROLLARY 3.3.** *Assume that  $A, B$  are coinfinite, semi-low<sub>1,5</sub> and have the prompt and low shrinking properties. Then there is an automorphism  $\Phi$  of  $\mathcal{E}$  such that  $\Phi(A) = B$ .*

**PROOF.** Since  $A, B$  are semi-low<sub>1,5</sub> and coinfinite, there exist according to [5] arrays  $(U_i)_{i \in N}, (\hat{V}_i)_{i \in N}, (\hat{U}_i)_{i \in N}, (V_i)_{i \in N}$  that satisfy the assumption of Theorem 3.2. In order to verify property (\*) one uses Lemma 4.5 of [5] in combination with the fact that if infinitely many elements are at some stage in state  $v = \langle e, \sigma, \tau \rangle$  then for every  $k > e$  there is a state  $v_k = \langle k, \tilde{\sigma}, \tilde{\tau} \rangle$  with  $\tilde{\sigma} \cap \{0, \dots, e\} = \sigma$  and  $\tilde{\tau} \cap \{0, \dots, e\} = \tau$  such that infinitely many of these elements are at some stage in state  $v_k$ .

COROLLARY 3.4 (SEE THEOREM 17 IN [4]). Assume,  $A, B$  are promptly simple and semi-low. Then there is an automorphism  $\Phi$  of  $\mathcal{E}$  such that  $\Phi(A) = B$ .

PROOF. This follows from Lemma 2.4, Lemma 2.5 and Corollary 3.3.

COROLLARY 3.5. There is a set  $A$  of low degree such that  $A$  is not promptly simple but  $A$  is automorphic to a promptly simple set of low degree (thus in particular to an r.e. generic set).

PROOF. Combine Theorem 2.7 with Corollary 3.3.

COROLLARY 3.6. There is a coinfinite set of low degree that has the splitting property but is not promptly simple.

PROOF. Take the set  $A$  from Theorem 2.7. According to Corollary 3.3 this set is automorphic to every semi-low promptly simple set. But every promptly simple set has the splitting property [7].

Note that one can prove this corollary as well without using automorphism results. It is not difficult to show directly that every semi-low set with the prompt shrinking property has the splitting property.

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