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ON THE ORBITS OF HYPERHYPERSIMPLE SETS

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Abstract. This paper contributes to the question of under which conditions recursively enumerable sets with isomorphic lattices of recursively enumerable supersets are automorphic in the lattice of all recursively enumerable sets. We show that hyperhypersimple sets (i.e. sets where the recursively enumerable supersets form a Boolean algebra) are automorphic if there is a Σ_3^0-definable isomorphism between their lattices of supersets. Lerman, Shore and Soare have shown that this is not true if one replaces Σ_3^0 by Σ_4^0.

§1. Introduction. For any subset S of the natural numbers N let $\mathscr{E}(S)$ be the lattice of sets $\{W \cap S \mid W$ recursively enumerable$\}$ under inclusion and let $\mathscr{E}^*(S)$ be the quotient lattice of $\mathscr{E}(S)$ modulo the ideal of finite subsets of S. One writes D^* for the equivalence class in $\mathscr{E}^*(S)$ containing $D \in \mathscr{E}(S)$. $\mathscr{E}$ and $\mathscr{E}^*$ are abbreviations for the lattice of all recursively enumerable (r.e.) sets $\mathscr{E}(N)$ and $\mathscr{E}^*(N)$, respectively. We write $\bar{A}$ for $N - A$. Observe that if A is r.e. then $\mathscr{E}(\bar{A})$ is trivially isomorphic to the lattice $\{W \mid W$ r.e. and $W \supseteq A\}$ of r.e. supersets of A.

An isomorphism Φ: $\mathscr{E}^*(S_1) \to \mathscr{E}^*(S_2)$ is called a Σ_n^0-isomorphism if there is a Σ_n^0-function h which maps N one-one onto N such that $\forall e \in N[\Phi((W_e \cap S_1)^*) = (W_{h(e)} \cap S_2)^*]$. This is obviously equivalent to the existence of Σ_n^0-functions f, g such that

$$\forall e \in N[\Phi((W_e \cap S_1)^*) = (W_{f(e)} \cap S_2)^* \wedge \Phi^{-1}((W_e \cap S_2)^*) = (W_{g(e)} \cap S)^*]$$

(see Soare [5]).

DEFINITION 1.1. An r.e. set A is called *hyperhypersimple* (hhs) if $\bar{A}$ is infinite and $\mathscr{E}(\bar{A})$ (or equivalently $\mathscr{E}^*(\bar{A})$) forms a Boolean algebra.

Hyperhypersimple sets were introduced as a strengthening of the notion of a simple set (i.e. sets which have a thin complement). An r.e. set is called simple if $\bar{A}$ is infinite and $A \cap W \neq \varnothing$ for every infinite r.e. set W. Lachlan [1] (see also Soare [7]) proved that an r.e. set A is hhs iff for every recursive function f such that the sets $W_{f(n)}$ are finite and pairwise disjoint there is some $n \in N$ s.t. $W_{f(n)} \subseteq A$ ($(W_e)_{e \in N}$ is a standard enumeration of the r.e. sets).

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We would like to introduce another characterization of hhs sets. In order to strengthen the definition of a simple set one can demand that A meets more sets than just the infinite r.e. sets, e.g. all infinite differences of two r.e. sets. There is of course a limitation: $\bar{A}$ itself is an infinite difference of two r.e. sets.

PROPOSITION 1.2. *An r.e. set A is hyperhypersimple if and only if $\bar{A}$ is infinite and $A \cap (V - W) \neq \varnothing$ for all r.e. sets V, W such that $V - W$ is infinite and not co-r.e.*

The proof is obvious.

Lachlan [1] proved that the Boolean algebras which occur as $\mathscr{E}^*(\bar{A})$ for hhs sets A are exactly those that have a Σ_3^0-representation. Soare [5] invented the automorphism machinery in order to show that r.e. sets A, B are automorphic (i.e. there is an automorphism Φ of $\mathscr{E}$ such that $\Phi(A) = B$, or equivalently, there is an automorphism Φ of $\mathscr{E}^*$ such that $\Phi(A^*) = B^*$) if $\mathscr{E}^*(\bar{A})$ and $\mathscr{E}^*(\bar{B})$ are the same *finite* Boolean algebras.

Later Lerman, Shore and Soare [2] produced examples of hhs sets A, B such that both $\mathscr{E}^*(\bar{A})$ and $\mathscr{E}^*(\bar{B})$ are the countable atomless Boolean algebra (i.e. A and B are "atomless hhs") but A and B are not automorphic. Obviously if A and B are atomless hhs then the standard back and forth construction yields a Σ_4^0-isomorphism between $\mathscr{E}^*(\bar{A})$ and $\mathscr{E}^*(\bar{B})$. Therefore Lerman, Shore and Soare [2] asked, "as a final attempt to generalize the maximal set automorphism result to hhs sets", whether hhs sets A and B are automorphic if there is a Σ_3^0-isomorphism between $\mathscr{E}^*(\bar{A})$ and $\mathscr{E}^*(\bar{B})$.

We give a positive answer to this question in Theorem 2.2. The proof of Theorem 2.2 follows the same outline as the proof for maximal sets in Soare [5]. We start with the given Σ_3^0-isomorphism between $\mathscr{E}^*(\bar{A})$ and $\mathscr{E}^*(\bar{B})$. Lerman, Shore and Soare [2] note that one can write every Σ_3^0-isomorphism as an isomorphism which is effective on suitable recursive arrays of r.e. sets $(X_n)_{n \in N}$ which form skeletons (i.e. $\forall e \exists n(W_e =^* X_n)$). We shrink the r.e. sets which are the values of the given isomorphism in such a way that one has still an isomorphism outside of A and B, but in addition the image sets have small enough intersection with A respectively B so that the "covering property" holds. According to Soare's extension theorem [5] one can then extend the isomorphism to an automorphism of $\mathscr{E}$ which maps A on B. In this shrinking process we first go to recursive arrays of r.e. sets $(U_e)_{e \in N}$ which are skeletons and where elements of $\bar{A}$ are almost always enumerated into these sets in the order of their indices: if $m < n$ then only finitely many elements of $\bar{A}$ are first enumerated in U_n and then in U_m (Lemma 2.3). We use at this point that A, B are hhs (actually the existence of a skeleton with the preceding properties implies that A is hhs). If A is hhs there exists for every r.e. set U_m a recursive set R such that $R \cap \bar{A} = U_m \cap \bar{A}$. We construct the array $(U_n)_{n \in N}$ in such a way that the index n of a set U_n with $n > m$ contains a guess at r.e. indices of R and $\bar{R}$ for a recursive set R with $R \cap \bar{A} = U_m \cap \bar{A}$ (besides this the index n contains guesses at certain values of the given Σ_3^0-isomorphism). If U_n does not guess correctly, we make it finite. If U_n guesses correctly, it can check for any number $x \in \bar{A}$ whether $x \in U_m$ or not. If U_n finds out that $x \in U_m$, it waits until x has appeared in U_m before it allows the enumeration of x in U_n.

M. Stob has pointed out that with this strategy one can actually (analogously as in Soare [5], Theorem 3.1) make sure that not only elements of $\bar{A}$ but also elements of $U_m \setminus A$ are almost always first enumerated in U_m before they come in U_n (for $m < n$).

We follow this suggestion because it simplifies the rest of the proof in an essential way.

In a second step (Lemma 2.4) we use the order preserving enumeration property of the constructed arrays in order to slow down the enumeration into sets in these arrays in such a way that the covering property is satisfied. This step requires more work than the corresponding step for maximal sets in Soare [5] (see the explanation at the beginning of the proof of Lemma 2.4).

Theorem 2.2 implies that all atomless hyperhypersimple sets with semi-low$_2$ complement are automorphic (Corollary 3.1). The problem of a complete characterization of the orbit of these sets is discussed in §3.

The following conventions and definitions will be used throughout the paper.

We demand from a simultaneous enumeration of an array of r.e. sets that at every stage at most one element is enumerated in one of the sets in the array (without repetitions).

For fixed enumerations $(A_s)_{s\in N}$ and $(B_s)_{s\in N}$ of r.e. sets A and B one defines $A \searrow B := \{x \mid \exists s(x \in A_s - B_s \wedge x \in B)\}$ and $A \setminus B := \{x \mid \exists s(x \in A_s - B_s)\}$.

For sets $A, B \subseteq N$ we write $A =^* B$ if A and B are equal except for finitely many numbers.

For any constructed r.e. set Z we write Z_s for the finite set of elements which have been enumerated in Z by the end of stage s.

For $e \geq 0$ and arrays $(X_n)_{n\in N}$ and $(Y_n)_{n\in N}$ we say that a number x has e-state $\langle e, \sigma, \tau\rangle$ (or simply: x has state $\langle e, \sigma, \tau\rangle$) w.r.t. $(X_n)_{n\in N}$ and $(Y_n)_{n\in N}$ if $\sigma = \{n \leq e \mid x \in X_n\}$ and $\tau = \{n \leq e \mid x \in Y_n\}$. We call triples $\langle e, \sigma, \tau\rangle$ with $\sigma, \tau \subseteq e + 1$ states (respectively, e-states). We reserve the letter ν for states. Following Soare [5] we say for states $\nu = \langle e, \sigma, \tau\rangle$, $\nu' = \langle e, \sigma', \tau'\rangle$ that $\nu \geq \nu'$ ("ν covers ν'") if $\sigma \supseteq \sigma'$ and $\tau \subseteq \tau'$.

§2. Hyperhypersimple sets with Σ_3^0-isomorphic lattices of supersets. We start with a technical lemma which will be used in the proof of Lemma 2.3. Since Σ_n^0-relations have the uniformization property, Π_n^0-relations cannot have the uniformization property as well. Lemma 2.1 shows that nevertheless a certain "weak uniformization property" holds for Π_n^0.

LEMMA 2.1. *Assume the relation $P \subseteq N \times N$ is Π_n^0-definable ($n \geq 1$). Then there is a Π_n^0-function f such that $\operatorname{dom} f = \{x \mid \exists y P(x, y)\}$ and $P(x, (f(x)_0))$ for every $x \in \operatorname{dom} f$.*

PROOF. Consider a Π_n^0-definition $P(x, y) \Leftrightarrow \forall u R(x, y, u)$ where R is Σ_{n-1}^0 (respectively recursive if $n = 1$). Then the following function f has the desired properties.

$$f(x) = \langle y, z\rangle :\Leftrightarrow \forall u R(x, y, u) \wedge \forall y' < y \exists u \leq z(\neg R(x, y', u))$$
$$\wedge\ \forall z' < z[\neg(\forall y' < y \exists u \leq z'(\neg R(x, y', u)))].$$

THEOREM 2.2. *Assume A and B are hyperhypersimple sets and there exists a Σ_3^0-isomorphism from $\mathscr{E}^*(\bar{A})$ onto $\mathscr{E}^*(\bar{B})$. Then there is an automorphism Φ of $\mathscr{E}$ with $\Phi(A) = B$.*

We first prove Lemma 2.3 and Lemma 2.4. These lemmata generalize Theorem 3.1 and Theorem 3.2, respectively, of Soare [5]. We show after Lemma 2.4 how

Theorem 2.2 is proved from these two lemmata together with Soare's extension theorem.

LEMMA 2.3. *Assume A and B are hyperhypersimple r.e. sets and Ψ is an isomorphism from $\mathscr{E}^*(\bar{A})$ onto $\mathscr{E}^*(\bar{B})$ with a Σ_3^0-function h that maps N one-one onto N such that, for every $e \in N$, $\Psi((W_e \cap \bar{A})^*) = (W_{h(e)} \cap \bar{B})^*$. Then there exist a strictly increasing function $t: N \to N$ and a simultaneous enumeration of A, B and r.e. sets $(U_n)_{n\in N}$, $(\tilde{V}_n)_{n\in N}$, $(\tilde{U}_n)_{n\in N}$, $(V_n)_{n\in N}$ such that*

(1) $U_{t(e)} = V_{t(e)} = W_e$ *for every* $e \in N$,

(2) $\tilde{V}_{t(e)} \cap \bar{A} = W_{h^{-1}(e)} \cap \bar{A}$ *and* $\tilde{U}_{t(e)} \cap \bar{B} = W_{h(e)} \cap \bar{B}$ *for every* $e \in N$,

(3) U_n, $\tilde{V}_n$, $\tilde{U}_n$, V_n *are finite if n is not in the range of t, and*

(4) *for every $m < n$ the sets*

$$U_n \searrow \tilde{V}_m, \quad \tilde{V}_n \searrow \tilde{V}_m, \quad (U_n \searrow U_m) \cap (U_m \setminus A),$$
$$\tilde{U}_n \searrow \tilde{U}_m, \quad V_n \searrow \tilde{U}_m, \quad (V_n \searrow V_m) \cap (V_m \setminus B)$$

are finite, and for every $m \leq n$ the sets

$$(\tilde{V}_n \searrow U_m) \cap (U_m \setminus A) \quad \text{and} \quad (\tilde{U}_n \searrow V_m) \cap (V_m \setminus B)$$

are finite.

PROOF. We define the desired function t inductively. For every $e \in N$ the value $t(e)$ contains (in coded form) $t \restriction e$, $h(e)$, $h^{-1}(e)$ and numbers e_j for $j \in \{0, \ldots, 7\}$ such that

$$\text{(a)} \quad \begin{array}{lll} W_{e_0} \cap \bar{A} = W_e \cap \bar{A}, & W_{e_0} \subseteq W_e, & \bar{W}_{e_0} = W_{e_1}, \\ W_{e_2} \cap \bar{A} = W_{h^{-1}(e)} \cap \bar{A}, & W_{e_2} \subseteq W_{h^{-1}(e)}, & \bar{W}_{e_2} = W_{e_3}, \\ W_{e_4} \cap \bar{B} = W_{h(e)} \cap \bar{B}, & W_{e_4} \subseteq W_{h(e)}, & \bar{W}_{e_4} = W_{e_5}, \\ W_{e_6} \cap \bar{B} = W_e \cap \bar{B}, & W_{e_6} \subseteq W_e, & \bar{W}_{e_6} = W_{e_7}. \end{array}$$

Obviously (a) implies that all the sets W_{e_j} are recursive. Indices e_j that satisfy (a) clearly exist for every $e \in N$ because A and B are hyperhypersimple. E.g. for every r.e. set W_e there is some r.e. set $W_{\tilde{e}}$ such that $\bar{W}_e \cap \bar{A} = W_{\tilde{e}} \cap \bar{A}$. We have then $W_e \cup (W_{\tilde{e}} \cup A) = N$, and by applying reduction to the sets W_e and $W_{\tilde{e}} \cup A$ we get r.e. sets W_{e_0}, W_{e_1} as desired.

One can express by a Π_2^0-formula that numbers $e_0, \ldots, e_7$ have property (a) relative to given indices e, $h(e)$, $h^{-1}(e)$. Since h is Σ_3^0 by assumption one can define a function t with all the mentioned properties by a Σ_3^0-formula (one uses here Σ_3^0-uniformization in order to choose numbers $e_0, \ldots, e_7$ that satisfy (a)). This is not yet quite enough because we need a Π_2^0-definition of t in order to get a simultaneous enumeration with property (4). For our strategy to get (4) it is essential that if index n guesses $t(e)$ correctly (and therefore $U_n = W_e$ etc.), the only indices $m < n$ where one of the sets U_m, $\tilde{V}_m$, $\tilde{U}_m$, V_m is infinite are those which guess $t(\tilde{e})$ correctly for some $\tilde{e} < e$. In this case we have $U_m = W_{\tilde{e}}$ etc., and index n knows the indices of recursive sets which coincide with U_m, $\tilde{V}_m$, $\tilde{U}_m$, V_m outside of A (respectively B) (we use here that $t \restriction e$ is contained in $t(e)$). For a Σ_3^0-definition of t there are in general many different witnesses for the first quantifier in the Σ_3^0-definition of t at argument e, and each of these witnesses gives rise to a different index n which guesses $t(e)$ correctly.

At this point we use Lemma 2.1. Define the function t by a Π_2^0-formula such that, for every $e \in N$, $t(e) = \langle (t(e))_0, \ldots, (t(e))_{12} \rangle$, the numbers $e_j := (t(e))_j$ for $j \leq 7$ satisfy (a), $(t(e))_8 = e$, $(t(e))_9 = t \upharpoonright e$, $(t(e))_{10} = h(e)$, $(t(e))_{11} = h^{-1}(e)$ and $(t(e))_{12}$ contains all the additional unique components of the value which arise through the application of Lemma 2.1 to those Σ_3^0-functions which were previously mentioned at the induction step of the definition of t (apply Lemma 2.1 to the Π_2^0-kernel of these Σ_3^0-definitions). Fix a Π_2^0-formula such that

$$\forall y \exists z R(e, n, y, z) \Leftrightarrow t(e) = n,$$

where R is recursive.

Fix a simultaneous enumeration of A, B and $(W_e)_{e \in N}$. Properties (1) and (2) are the positive requirements which have to be satisfied by our simultaneous enumeration of A, B and r.e. sets $(U_n)_{n \in N}$, $(\tilde{V}_n)_{n \in N}$, $(\tilde{U}_n)_{n \in N}$, $(V_n)_{n \in N}$. We specify in the following several restraints to the enumeration which ensure simultaneously the other properties.

In order to satisfy property (3) we

(b) enumerate at stage s a kth element into

$$U_n(\tilde{V}_n, \tilde{U}_n, V_n) \quad \text{only if} \quad \forall y \leq k \exists z \leq s\, R((n)_8, n, y, z).$$

Further we enumerate a number into U_n only after it has appeared in $W_{(n)_0}$ or in our new enumeration of A. Similarly we enumerate a number into V_n only after it has appeared in $W_{(n)_6}$ or in B. This restriction is harmless because $W_e \subseteq W_{(t(e))_0} \cup A$ and $W_e \subseteq W_{(t(e))_6} \cup B$. We get then

(c) for every $e \in N$

$$U_{t(e)} \setminus A \subseteq W_{(t(e))_0} \quad \text{and} \quad V_{t(e)} \setminus B \subseteq W_{(t(e))_6}.$$

In addition we never enumerate a number into $\tilde{V}_n(\tilde{U}_n)$ after it has appeared in $A(B)$. This restriction and property (c) will help to satisfy (4).

The following restriction is essential for the satisfaction of (4). We enumerate a number into U_n only if $(n)_9$ is a function g from e into N, where $e := (n)_8$. Before we put any $x \in W_e$ into U_n we wait until, for every $i < e$, x has appeared in $W_{(g(i))_0}$ or $W_{(g(i))_1}$ and in $W_{(g(i))_2}$ or $W_{(g(i))_3}$. After this has occurred we wait a little longer until x has been enumerated in $U_{g(i)}$ for every $i < e$, where x has appeared in $W_{(g(i))_0}$, and until x has been enumerated in $\tilde{V}_{g(i)}$ or in A for every $i < e$, where x has appeared in $W_{(g(i))_2}$.

We set up the same restrictions for the enumeration of numbers $x \in W_{(n)_2}$ into $\tilde{V}_n$. But in addition we wait until x has appeared in $W_{(n)_0}$ or in $W_{(n)_1}$, and if x appears in $W_{(n)_0}$ we wait until x has been enumerated in U_n (this is necessary in order to keep $\tilde{V}_n \setminus U_n$ small, as it is demanded in property (4)).

The restrictions for the B-side ($\tilde{U}_n$ and V_n) are symmetrical.

In a dovetail fashion we enumerate for all $n \in N$ every element of $W_{(n)_8}$ ($W_{(n)_2}$, $W_{(n)_4}$, $W_{(n)_8}$) into $U_n(\tilde{V}_n, \tilde{U}_n, V_n)$ as soon as it is not anymore prohibited by some restriction.

One easily verifies properties (1) and (2) simultaneously by induction on e.

In order to prove (4) we fix some m, n with $m \leq n$. Because of (b) we can assume that $n = t(e)$ and $m = t(i)$ for some $i \leq e$.

Assume first that $m < n$ and x is first enumerated into U_n and later into $\tilde{V}_m$. According to the restrictions, x is required to appear in $W_{(t(i))_2}$ or $W_{(t(i))_3}$ before it comes into U_n. $x \in W_{(t(i))_3}$ is impossible if $x \in \tilde{V}_m \subseteq W_{(t(i))_2} = \bar{W}_{(t(i))_3}$. Therefore $x \in W_{(t(i))_2}$ and the enumeration of x into U_n is delayed until x has appeared in $\tilde{V}_m$ or in A.

Assume now that $m \leq n$ and x is first enumerated into $\tilde{V}_n$ and later into U_m. We show that $x \notin U_m \setminus A$. Before the enumeration of x in $\tilde{V}_n$, x has to appear in $W_{(t(i))_0}$ or $W_{(t(i))_1}$. We cannot have $x \in W_{(t(i))_0}$, because then x can only be enumerated into $\tilde{V}_n$ after it has appeared in U_m. Therefore $x \in W_{(t(i))_1}$, and this implies $x \notin U_m \setminus A$ according to (c).

The rest is proved analogously. Note that, although we have used a different terminology, this proof is an extension of Soare's Theorem 3.1 in [5].

LEMMA 2.4. *Assume A and B are simple r.e. sets. Further assume that there is a simultaneous enumeration of A, B and r.e. sets $(U_n)_{n\in N}, (\tilde{V}_n)_{n\in N}, (\tilde{U}_n)_{n\in N}, (V_n)_{n\in N}$ which satisfies property* (4) *of Lemma* 2.3 *and such that for every state v*

$$\{x \in \bar{A} \mid x \text{ has state } v \text{ w.r.t. } (U_n)_{n\in N}, (\tilde{V}_n)_{n\in N}\} \text{ is infinite}$$
$$\Leftrightarrow \{x \in \bar{B} \mid x \text{ has state } v \text{ w.r.t. } (\tilde{U}_n)_{n\in N}, (V_n)_{n\in N}\} \text{ is infinite.}$$

Then there is a simultaneous enumeration of $A, B, (U_n)_{n\in N}, (V_n)_{n\in N}$ and r.e. sets $(\hat{V}_n)_{n\in N}$ and $(\hat{U}_n)_{n\in N}$ such that

(I) *for every $n \in N$*

$$\hat{V}_n \cap \bar{A} =^* \tilde{V}_n \cap \bar{A} \quad \text{and} \quad \hat{U}_n \cap \bar{B} =^* \tilde{U}_n \cap \bar{B},$$

(II) *for every $n \in N$*

$$A \searrow \hat{V}_n = \varnothing \quad \text{and} \quad B \searrow \hat{U}_n = \varnothing,$$

(III) (*the covering property*): *for every v such that*

$$D^B_v := \{\hat{x} \mid \exists s(\hat{x} \in B_{s+1} - B_s \wedge \hat{x} \text{ has state } v \text{ w.r.t. } (\hat{U}_{n,s})_{n\in N}, (V_{n,s})_{n\in N})\}$$
is infinite,

there is some $v' \geq v$ such that

$$D^A_{v'} := \{x \mid \exists s(x \in A_{s+1} - A_s \wedge x \text{ has state } v \text{ w.r.t. } (U_{n,s})_{n\in N}, (\hat{V}_{n,s})_{n\in N})\}$$
is infinite,

and for every state v' such that $D^A_{v'}$ is infinite there is some $v \leq v'$ such that D^B_v is infinite.

PROOF. We make $\hat{U}_i \subseteq \tilde{U}_i$ and $\hat{V}_i \subseteq \tilde{V}_i$ for every $i \in N$. The idea is to slow down the enumeration of elements of $\tilde{U}_i$ into $\hat{U}_i$ in such a way that a number $\hat{x}$ gets into state v w.r.t. $(\hat{U}_i)_{i\in N}, (V_i)_{i\in N}$ only if we have already seen some $y \in D^A_{\tilde{v}}$ for some $\tilde{v} \geq v$. Thus in case that $\hat{x}$ falls later in state v into B (so that $\hat{x} \in D^B_v$), we have already some covering element y on the A-side in store. The problem remains, *which* of the $\hat{U}_i$ with $\hat{x} \in \hat{U}_i$ should receive the duty to prevent $\hat{x}$ from getting into $\hat{U}_i$ until this covering element y has appeared. But since we have enumeration in order (property (4)), it is enough to wait for the covering element y as above before we enumerate $\hat{x}$ into $\hat{U}_i$, where $i \leq |v|$ is maximal with $\hat{x} \in \tilde{U}_i$. This strategy gives rise to condition (c) in the following construction. We verify in Claim 3 that the slowing down of the enumeration according to condition (c) yields the desired covering property (III).

We have to justify as well, that condition (c) does not prevent too many elements of $\tilde{U}_i \cap \bar{B}$ from getting into $\hat{U}_i$. We have to make sure that $(\hat{U}_i \cap \bar{B})^*$ is still the image of $(U_i \cap \bar{A})^*$ under the considered Σ_3^0-isomorphism from $\mathscr{E}^*(\bar{A})$ onto $\mathscr{E}^*(\bar{B})$. This is more difficult than in the analogous construction for maximal sets in Soare [5] (Theorem 3.2). In Soare's situation the existence of the covering element y is automatically guaranteed because there the considered array has the additional property that, for every $i \in N$, $\bar{V}_i \cap \bar{B} \neq \varnothing \Leftrightarrow V_i \cap \bar{B}$ is finite. In our situation we have to actively slow down the enumeration on the A-side in order to force the appearance of the desired covering element y. Thus the clerk for the enumeration into sets $\hat{V}_i$ goes on strike until his demand for a certain covering element has been satisfied. During the strike the clerk puts elements on waiting lists L_ν, and he refuses to enumerate an element on a waiting list L_ν into any set $\hat{V}_i$ with $i > |\nu|$ (this is condition (b) in the construction below). The clerk takes elements off list L_ν if his demand for a certain covering element has been satisfied. His demand is almost always satisfied, because the elements which are put on a certain waiting list form an r.e. set. Therefore if one puts infinitely many elements on a certain list, some element on this list gets enumerated into the simple set A. During the time this element was on the list, it has not been enumerated in too many sets $\hat{V}_i$. Therefore it has all the properties of the desired covering element y.

We now describe the exact construction. We show afterwards in Claim 2 and Claim 3 that the constructed simultaneous enumeration has the desired properties. Claim 1 is needed for the proof of Claim 2. In the following enumeration sometimes several elements are enumerated at one stage. One can easily repair this by distributing the action at one stage over several (new) stages.

CONSTRUCTION. *Stage* $s \geq 0$. If in the given simultaneous enumeration some element is enumerated at stage s into one of the sets A, B, $(U_i)_{i \in N}$ or $(V_i)_{i \in N}$, we enumerate this element now in our new enumeration into the same set.

If $\hat{x}$ was on list $\hat{L}_\nu$ with $\nu = \langle |\nu|, \sigma, \tau \rangle$ at the end of stage $s-1$, then $\hat{x}$ is taken off $\hat{L}_\nu$ if at stage s some $\hat{y} \geq \hat{x}$ is enumerated into B in state $\langle \hat{x}, \sigma_{\hat{y}}, \tau_{\hat{y}} \rangle$ w.r.t. $(\hat{U}_{i,s-1})_{i \in N}$, $(V_{i,s-1})_{i \in N}$ with $\sigma_{\hat{y}} \subseteq \sigma$, $\tau \subseteq \tau_{\hat{y}}$ and $|\nu| + 1 \in \tau_{\hat{y}}$.

If $\hat{x}$ was on list $\hat{L}_\nu$ at the end of stage $s-1$ and $\hat{x} \in \tilde{U}_{i,s} - \tilde{U}_{i,s-1}$ for some $i \leq |\nu|$, then we cancel $\hat{x}$ from $\hat{L}_\nu$.

$\hat{x}$ is enumerated in $\hat{U}_n$ at stage s if the following three conditions are satisfied.

(a) $\hat{x} \notin B$, $\hat{x} \in \tilde{U}_{n,s-1} - \hat{U}_{n,s-1}$ and, for every $m < n$, $\hat{x} \in \tilde{U}_{m,s-1} \Leftrightarrow \hat{x} \in \hat{U}_{m,s-1}$.

(b) $\hat{x}$ is at the moment on no list $\hat{L}_\nu$ with $|\nu| < n$.

(c) Some element $y \geq \hat{x}$ was enumerated into A at some stage $t < s$ in $\hat{x}$-state ν_y w.r.t. $(U_{i,t})_{i \in N}$, $(\hat{V}_{i,t})_{i \in N}$ and $\nu_y \geq \nu$, where ν is the $\hat{x}$-state of $\hat{x}$ w.r.t. the arrays $\hat{U}_{0,s-1}, \ldots, \hat{U}_{n-1,s-1}, N, \varnothing, \varnothing, \ldots$ and $(V_{i,s-1})_{i \in N}$.

Finally we put some element $\hat{x}$ on list $\hat{L}_\nu$ at stage s if $\hat{x} > |\nu|$, $\hat{x} \notin B_s$, $\hat{x} \in V_{|\nu|+1,s} - V_{|\nu|+1,s-1}$, $\hat{x} \notin \tilde{U}_{|\nu|+1,s}$ and ν is the state of $\hat{x}$ w.r.t. $(U_{i,s})_{i \in N}$, $(V_{i,s})_{i \in N}$.

We proceed at stage s analogously on the A-side. This ends the construction.

CLAIM 1. (a) *Only finitely many elements are cancelled from each list.*
(b) *If, for some list $\hat{L}_\nu$, $\hat{K}_\nu := \{\hat{x} \mid \hat{x}$ is put on list $\hat{L}_\nu$ during the construction$\}$ is infinite, then every element of $\hat{K}_\nu$ either is taken off $\hat{L}_\nu$ or is cancelled from $\hat{L}_\nu$. The analogous fact holds for the lists L_ν.*

PROOF OF CLAIM 1. (a) If infinitely many elements are cancelled from list $\hat{L}_\nu$ then we have $V_{|\nu|+1} \searrow \tilde{U}_i$ infinite for some $i \leq |\nu|$. This contradicts property (4) of Lemma 2.3.

(b) Assume $\hat{K}_v$ is infinite. Assume that some element $\hat{x}_0$ is put on list $\hat{L}_v$ at stage t_0 and $\hat{x}_0$ is never taken off $\hat{L}_v$ or is cancelled from $\hat{L}_v$. Define

$$K := \{\hat{x} > \hat{x}_0 \mid \hat{x} \text{ is put on list } \hat{L}_v \text{ at some stage } t_{\hat{x}} > t_0,$$
$$\hat{x} \notin \hat{U}_{i,t_{\hat{x}}} \text{ for } |v| < i \leq \hat{x}_0 \text{ and } \hat{x} \text{ is not cancelled from } \hat{L}_v\}.$$

K is r.e. because of part (a). Further, for every i with $|v| < i \leq \hat{x}_0$ we have $(\tilde{U}_i \searrow V_{|v|+1}) \cap (V_{|v|+1} \backslash B)$ finite by assumption. Since $x \notin \tilde{U}_{i,t_{\hat{x}}-1}$ implies $x \notin \hat{U}_{i,t_{\hat{x}}}$ and since $\hat{K}_v \subseteq V_{|v|+1} \backslash B$, we get then that almost all elements of $\hat{K}_v$ are in K. Therefore K is infinite. Since B is simple, there exists some $\hat{y} \in K$ which is enumerated in B at some stage $s > t_{\hat{x}} > t_0$. $\hat{y}$ is not enumerated in any $\hat{U}_i$ with $i > |v|$ as long as it is on list $\hat{L}_v$ and $\hat{y}$ is not cancelled from list $\hat{L}_v$. Therefore $\hat{y}$ has $\hat{x}_0$-state $\langle \hat{x}_0, \sigma_{\hat{y}}, \tau_{\hat{y}} \rangle$ w.r.t. $(\hat{U}_{n,s-1})_{n \in N}, (V_{n,s-1})_{n \in N}$ with $\sigma_{\hat{y}} \subseteq \sigma, \tau \subseteq \tau_{\hat{y}}$ and $|v| + 1 \in \tau_{\hat{y}}$, where $v := \langle |v|, \sigma, \tau \rangle$. Then $\hat{x}_0$ is taken off $\hat{L}_v$ at stage s, a contradiction.

CLAIM 2. *Assume that for $v = \langle n, \sigma, \tau \rangle$ the set $\hat{S}_v := \{\hat{x} \in \bar{B} \mid \hat{x}$ has state v w.r.t. $(\tilde{U}_i)_{i \in N}, (V_i)_{i \in N}\}$ is infinite. Then almost all elements of $\hat{S}_v$ have state v w.r.t. $(\hat{U}_i)_{i \in N}, (V_i)_{i \in N}$.*

The analogous fact holds for $\bar{A}$.

PROOF OF CLAIM 2. We proceed by induction on n. It is obvious that, for every $i \in N$, $\tilde{V}_i \subseteq \hat{V}_i$ and $\tilde{U}_i \subseteq \hat{U}_i$. Assume that $\hat{S}_v$ is infinite. By the induction hypothesis almost all elements of $\hat{S}_v$ have state $\langle n - 1, \sigma \cap n, \tau \cap n \rangle$ w.r.t. $(\hat{U}_i)_{i \in N}, (V_i)_{i \in N}$. We have to prove something only if $n \in \sigma$, in which case we show that almost all elements of $\hat{S}_v$ are enumerated in $\hat{U}_n$. Because of Claim 1 only finitely many elements of $\hat{S}_v$ can be prevented from enumeration in $\hat{U}_n$ by condition (b). Therefore condition (c) is the only serious obstacle.

By the assumption of this lemma the set $S_v := \{x \in \bar{A} \mid x$ has state v w.r.t. $(U_i)_{i \in N}, (\tilde{V}_i)_{i \in N}\}$ is infinite as well. Almost all elements of S_v are put on list L_v, because the given enumeration satisfies property (4) of Lemma 2.3. Therefore infinitely many elements are taken off list L_v according to Claim 1. Consider then some $\hat{x} \in \hat{S}_v$. Take some $x \geq \hat{x}$ which is taken off L_v at stage t because at stage t some $y \geq x$ is enumerated in A in state $\langle x, \sigma_y, \tau_y \rangle$ w.r.t. $(U_{i,t-1})_{i \in N}, (\hat{V}_{i,t-1})_{i \in N}$ with $\sigma \subseteq \sigma_y$ and $\tau_y \subseteq n$. This enumeration of y in A at stage t supplies the desired witness for $\hat{x}$ to satisfy condition (c) at stages $s > t$.

CLAIM 3. *The new enumeration satisfies the covering property* (III).

PROOF OF CLAIM 3. Assume that D_v^B is infinite, where $v = \langle n, \sigma, \tau \rangle$. We consider first the case where $\sigma \neq \emptyset$. Let n' be the maximal element of σ. Fix a natural number m. We show that some $D_{\tilde{v}}^A$ with $\tilde{v} \geq v$ contains an element $y \geq m$. Take some element $\hat{x} \geq m, n$ of D_v^B with $\hat{x} \notin \hat{U}_{n'} \searrow \tilde{U}_i$ for all $i < n'$. For the stage s where $\hat{x}$ is enumerated into $\hat{U}_{n'}$ we have $\hat{x} \notin A_s$, and we know because of condition (c) that there exist some $y \geq \hat{x}$ and a stage $t < s$ such that y was enumerated in A at stage t in n-state v_y w.r.t. $(U_{i,t})_{i \in N}, (\hat{V}_{i,t})_{i \in N}$ and $v_y \geq v'$, where v' is the n-state of $\hat{x}$ with respect to $\hat{U}_{0,s-1}, \ldots, \hat{U}_{n'-1,s-1}, N, \emptyset, \emptyset, \ldots$ and $(V_{i,s-1})_{i \in N}$ (we use here that $\hat{x} \geq n$). Because of condition (a) we have $\hat{x} \in U_{n',s-1}$ and, for all $i < n'$, $\hat{x} \in \tilde{U}_{i,s-1} \Leftrightarrow \hat{x} \in \hat{U}_{i,s-1}$. Further, by our choice of $\hat{x}$ this element is not enumerated into any $\tilde{U}_i$ with $i < n'$ after stage $s - 1$. Therefore $\hat{x}$ is not enumerated into any new $\hat{U}_i$ with $i \leq n$ until it enters B in state v at some later stage. This implies that $v' \geq v$ and therefore $y \in D_{\tilde{v}}^A$ for $\tilde{v} := v_y \geq v$.

Concerning the case $\sigma = \emptyset$ we assume without loss of generality that there is an infinite recursive set $R \subseteq A$ such that $R \cap \tilde{V}_i = \emptyset$ for all $i \in N$. Then every element of R is in some $D^A_{\tilde{v}}$ with $\tilde{v} \geq v$.

The other half of the covering property is proved analogously.

PROOF OF THEOREM 2.2. Let Ψ be an isomorphism from $\mathscr{E}^*(\bar{A})$ onto $\mathscr{E}^*(\bar{B})$ with a Σ^0_3-function h which maps N one-one onto N such that $\Psi((W_e \cap \bar{A})^*) = (W_{h(e)} \cap \bar{B})^*$ for every $e \in N$. We get then a simultaneous enumeration of A, B and r.e. sets $(U_n)_{n\in N}$, $(\tilde{V}_n)_{n\in N}$, $(\tilde{U}_n)_{n\in N}$, $(V_n)_{n\in N}$ which satisfy properties (1)–(4) of Lemma 2.3. Since Ψ is an isomorphism, the properties (1), (2) and (3) imply that

$$(*) \qquad \{x \in \bar{A} \mid x \text{ has state } v \text{ w.r.t. } (U_n)_{n\in N}, (\tilde{V}_n)_{n\in N}\} \text{ is infinite}$$
$$\Leftrightarrow \{\hat{x} \in \bar{B} \mid \hat{x} \text{ has state } v \text{ w.r.t. } (\tilde{U}_n)_{n\in N}, (V_n)_{n\in N}\} \text{ is infinite}$$

for every state v. Therefore all assumptions of Lemma 2.4 are satisfied and we get a simultaneous enumeration of A, B, $(U_n)_{n\in N}$, $(V_n)_{n\in N}$ together with r.e. sets $(\hat{V}_n)_{n\in N}$, $(\hat{U}_n)_{n\in N}$ which satisfy properties (I)–(III) of Lemma 2.4. According to Soare [5], property (I) together with (*) implies the existence of a function q which maps $\bar{A}$ one-one onto $\bar{B}$ such that, for every $n \in N$, $q[U_n \cap \bar{A}] =^* \hat{U}_n \cap \bar{B}$ and $q^{-1}[V_n \cap \bar{B}] =^* \hat{V}_n \cap \bar{A}$ (see the construction of Theorem 1.3 in Soare [5] or the end of the proof of Theorem 1.2 in Maass [3]).

Further, properties (II) and (III) are just the assumptions of Soare's extension theorem (Theorem 2.2 in Soare [5]). According to the extension theorem there exist a function p and recursive arrays of r.e. sets $(V_n^-)_{n\in N}$, $(U_n^-)_{n\in N}$ such that p maps A one-one onto B and, for every $n \in N$,

$$p[U_n \cap A] =^* (\hat{U}_n \cap B) \cup U_n^- \quad \text{and} \quad p^{-1}[V_n \cap B] =^* (\hat{V}_n \cap A) \cup V_n^- .$$

Consider the function $r := q \cup p$. r maps N one-one onto N, $r[A] = B$ and for every $n \in N$ we have $r[U_n] =^* \hat{U}_n \cup U_n^-$ and $r^{-1}[V_n] =^* \hat{V}_n \cup V_n^-$. According to property (1) we have $W_e = U_{t(e)} = V_{t(e)}$ for every $e \in N$. Together this implies that for every $e \in N$ the sets $r[W_e]$ and $r^{-1}[W_e]$ are r.e.. Therefore the function Φ defined by $\Phi(W_e^*) := r[W_e]^*$ for every $e \in N$ is an automorphism of $\mathscr{E}^*$ with $\Phi(A^*) = B^*$.

REMARK 2.5. It is easy to see that the automorphism which is constructed in the proof of Theorem 2.2 is a Σ^0_3-automorphism.

§3. Some remarks on atomless hyperhypersimple sets. A hyperhypersimple set A is called *atomless* if it has no maximal superset, or equivalently, if $\mathscr{E}^*(\bar{A})$ is the countable atomless Boolean algebra.

According to Martin [4] every hyperhypersimple set is of high degree. But there are many hyperhypersimple sets A where $\bar{A}$ is semi-low$_2$ (i.e. $\{e \mid W_e \cap \bar{A}$ is infinite$\} \leq_T 0''$; see Soare [6]). For example, all maximal sets have this property. In addition the standard construction of an atomless hyperhypersimple set (Lachlan [1]) automatically produces a set A where $\bar{A}$ is semi-low$_2$.

COROLLARY 3.1. *If A, B are atomless hyperhypersimple sets with semi-low$_2$ complement then there is an automorphism Φ of $\mathscr{E}$ such that $\Phi(A) = B$.*

PROOF. Since $\bar{A}$, $\bar{B}$ are semi-low$_2$, the standard back and forth construction of an isomorphism between the countable atomless Boolean algebras $\mathscr{E}^*(\bar{A})$ and $\mathscr{E}^*(\bar{B})$ produces a Σ^0_3-isomorphism. Therefore A and B are automorphic according to Theorem 2.2.

A challenging open problem is the complete characterization of the orbit $\mathscr{X}$ of atomless hyperhypersimple sets with semi-low$_2$ complement, i.e.

$$\mathscr{H} := \{B \in \mathscr{E} \mid \text{there exist an automorphism } \Phi \text{ of } \mathscr{E} \text{ and an atomless hyperhypersimple set } A \text{ with } \bar{A} \text{ semi-low}_2 \text{ such that } \Phi(A) = B\}.$$

Of course every set in $\mathscr{H}$ is atomless hyperhypersimple. If the orbit $\mathscr{H}$ contains sets A with $\bar{A}$ not semi-low$_2$, the existing methods for the construction of automorphisms will not suffice for a proof of this fact. All existing methods produce Σ_3^0-automorphisms, and the property "$\bar{A}$ semi-low$_2$" is invariant under Σ_3^0-automorphisms. This leads to a more general open question: Is "$\bar{A}$ semi-low$_2$" invariant under all automorphisms of $\mathscr{E}$?

On the other hand Lerman, Shore and Soare [2] have shown that some atomless hyperhypersimple sets are not in the orbit $\mathscr{H}$. According to [2] a set $B \subseteq A$ is called an r-maximal major subset of A (written $B \subseteq_{rm} A$) if B is a major subset of A and $A - B \subseteq^* R \vee A - B \subseteq^* \bar{R}$ for every recursive set R. Obviously the property "A has an r-maximal major subset" is definable over $\mathscr{E}$ and therefore invariant under automorphisms. Lerman, Shore and Soare [2] construct an atomless hyperhypersimple set which has no r-maximal major subset. Further they show that every atomless hyperhypersimple set with semi-low$_2$ complement has an r-maximal major subset.

We show below that not all atomless hyperhypersimple sets with r-maximal major subsets are automorphic and therefore the class of these sets does not coincide with $\mathscr{H}$. We do this by producing a stronger $\mathscr{E}$-definable property ("A has everywhere r-maximal major subsets") which divides this class. It will be obvious that if A is in $\mathscr{H}$ then A has everywhere r-maximal major subsets. On the other hand we produce in Proposition 3.5 a set A where $\bar{A}$ is not semi-low$_2$ but which has all known $\mathscr{E}$-definable properties of sets in $\mathscr{H}$ (including our new property). In order to characterize the orbit $\mathscr{H}$ one will have to determine whether or not this set A is in $\mathscr{H}$. This may be very hard. If $A \in \mathscr{H}$ there are the previously mentioned difficulties in proving this because $\bar{A}$ is not semi-low$_2$. If $A \notin \mathscr{H}$ one might be lucky and find a new $\mathscr{E}$-definable property which separates A from sets in $\mathscr{H}$. But it might well be the case that A has the same 1-type as sets in $\mathscr{H}$, so that $A \notin \mathscr{H}$ has to be proved by different methods.

From the definition of an r-maximal major subset it is clear that every set $B \subseteq_{rm} A$ defines an ultrafilter $U_b := \{(R \cap \bar{A})^* \mid R \text{ recursive and } A - B \subseteq^* R\}$ on the Boolean algebra $\{(R \cap \bar{A})^* \mid R \text{ recursive}\}$.

DEFINITION 3.2. *A has everywhere r-maximal major subsets* if for every recursive set R such that $(R \cap \bar{A})^* \neq \phi^*$ there exists a set $B \subseteq_{rm} A$ with $(R \cap \bar{A})^* \in U_B$ (i.e. $A - B \subseteq^* R$).

Lerman, Shore and Soare [2] introduce the notion of a preference function for A in order to get information about r-maximal major subsets of A. We relativize their definition to a given recursive set R.

DEFINITION 3.3. Let $R_i := \{x \mid \forall y \leq x(\phi_i(y)\downarrow \wedge \phi_i(x) = 1)\}$ for a standard enumeration $(\phi_i)_{i \in N}$ of the partial recursive functions, and let A be a simple set. For a given recursive set R we say that a $\{0, 1\}$-valued function h is a *preference function* for

A which concentrates on R if, for every initial segment τ of h, $R \cap R_\tau \cap \bar{A}$ is infinite, where $R_\tau := \bigcap\{R_i \mid \tau(i) = 1\} \cap \bigcap\{\bar{R}_i \mid \tau(i) = 0\}$.

The proof of Theorem 1.2 in Lerman, Shore, Soare [2] shows that for every simple set A and for every recursive set R with $(R \cap \bar{A})^* \neq \varnothing^*$ there exists a set $B \subseteq_{rm} A$ with $A - B \subseteq^* R$ if and only if there is a Δ_3^0-preference function for A which concentrates on R.

If $\bar{A}$ is semi-low$_2$, then for every recursive set R with $(R \cap \bar{A})^* \neq \varnothing^*$ there is a Δ_3^0-preference function for A which concentrates on R. Therefore all sets in the orbit $\mathscr{H}$ have everywhere r-maximal major subsets.

PROPOSITION 3.4. *There is an atomless hyperhypersimple set A such that A has an r-maximal major subset but A has not everywhere r-maximal major subsets.*

PROOF. Let C be an atomless hyperhypersimple set with $\bar{C}$ semi-low$_2$, and let D be an atomless hyperhypersimple set which has no r-maximal major subset (such a D exists according to [2]). Fix a recursive enumeration f of C and define $A := f[D]$. According to Lachlan [1], $A \cup \bar{C}$ is then r.e. and thus A is atomless hyperhypersimple. Consider some recursive set R such that $R \cap \bar{A} = C \cap \bar{A}$. Obviously A has a Δ_3^0-preference function which concentrates on $\bar{R}$ (since $\bar{C}$ is semi-low$_2$), but A has no Δ_3^0-preference function which concentrates on R (since D has no Δ_3^0-preference function).

PROPOSITION 3.5. *There exists an atomless hyperhypersimple set A such that A has everywhere r-maximal major subsets but $\bar{A}$ is not semi-low$_2$.*

PROOF. Consider a Boolean algebra $\mathfrak{B}$ and a sequence $(b_i)_{i \in N}$ of elements of $\mathfrak{B}$ such that every member of $\mathfrak{B}$ can be obtained from members in the sequence by the operations of union, intersection and complementation. Associated with $\mathfrak{B}$ and $(b_i)_{i \in N}$ there is a function $F: 2^{<\omega} \to \{0, 1\}$ defined by $F(\sigma) = 0 :\Leftrightarrow b_\sigma =_{\mathfrak{B}} 0_{\mathfrak{B}}$, where $b_\sigma := \bigcap\{b_i \mid \sigma(i) = 1\} \cap \bigcap\{\bar{b}_i \mid \sigma(i) = 0\}$. If F is Σ_3^0 then there exists according to Lachlan [1] a hyperhypersimple set A with an isomorphism $\Psi: \mathfrak{B} \to \mathscr{E}^*(\bar{A})$. Lerman, Shore and Soare [2] point out that Lachlan's construction produces in addition Δ_3^0-functions f, g such that for every $i \in N$

$$\Psi(b_i) = (W_{f(i)} \cap \bar{A})^* \quad \text{and} \quad \Psi^{-1}((W_i \cap \bar{A})^*) = \bigcup\{b_\sigma \mid \sigma \in g(i)\}$$

(for every $i \in N$, $g(i)$ is a finite subset of $2^{<\omega}$).

We fix now a Σ_3^0 set $S \in \mathbf{0}'''$ and consider the Boolean algebra $\mathfrak{B}$ with generators $(b_i)_{i \in N}$ such that the associated function F satisfies

$$F(\sigma) = 0 \Leftrightarrow \exists i \leq \operatorname{lth} \sigma (i \in S \wedge \sigma(i) = 1).$$

Obviously F is Σ_3^0. We can picture this Boolean algebra $\mathfrak{B}$ by a binary tree with the branching at level i representing intersection with b_i and $\bar{b}_i$ respectively. We get the binary tree for $\mathfrak{B}$ by chopping off from the full binary tree all nodes on a branch to the left which starts at a node on level i with $i \in S$. Since $\bar{S}$ is infinite there are still infinitely many levels where the branching is preserved. Therefore $\mathfrak{B}$ is the countable atomless Boolean algebra.

Let A be the associated hyperhypersimple set with functions Ψ, f, g as above. Then, for every $i \in N$, $i \in S \Leftrightarrow b_i =_{\mathfrak{B}} 0_{\mathfrak{B}} \Leftrightarrow \Psi(b_i) = \phi^* \Leftrightarrow W_{f(i)} \cap \bar{A}$ finite, and therefore $\bar{A}$ semi-low$_2$ would imply that $S \leq_T \mathbf{0}''$.

Finally, fix some recursive set R with $R \cap \bar{A}$ infinite. Obviously there exists some $\sigma_0 \in 2^{<\omega}$ such that $0_{\mathfrak{B}} <_{\mathfrak{B}} b_{\sigma_0} \leq_{\mathfrak{B}} \Psi^{-1}((R \cap \bar{A})^*)$.

We now define a Δ_3^0-preference function h for A which concentrates on R. We can do this because in the previously described binary tree branches to the right are never chopped off. Thus $0_{\mathfrak{B}} <_{\mathfrak{B}} b_\sigma \leq_{\mathfrak{B}} b_{\sigma_0}$ if σ has the form $\sigma_0 * 0 * \cdots * 0$. For a given $e \in N$ we first go (in a Δ_3^0 way) to an index $\tilde{e} \in N$ such that $R_e = W_{\tilde{e}}$. Then we consider all sequences $\sigma \in g(\tilde{e})$. If there is some $\sigma \in g(\tilde{e})$ of the form $\sigma = \sigma_0 * 0 * \cdots * 0$ or such that $\sigma \subseteq \sigma_0$ we set $h(e) = 1$. Otherwise we set $h(e) = 0$.

It is obvious that in the case where $h(e) = 0$ there is some σ of the form $\sigma_0 * 0 * \cdots * 0$ with $b_\sigma \leq_{\mathfrak{B}} \Psi^{-1}((R \cap \bar{R}_e \cap \bar{A})^*)$. Therefore for every initial segment $\tau \subset h$ there is some σ of the form $\sigma_0 * 0 * \cdots * 0$ such that $\Psi^{-1}((R \cap R_\tau \cap \bar{A})^*) \geq_{\mathfrak{B}} b_\sigma >_{\mathfrak{B}} 0_{\mathfrak{B}}$. Thus h is a Δ_3^0-preference function for A which concentrates on R.

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