Additional material to the paper: What can a Neuron Learn with Spike-Timing-Dependent Plasticity?

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1 Full proof of Proposition 4.1

Proposition 4.1: *There exists a matrix* $C = \{c_{ij}\}_{i,j=1,\ldots,n}$ *with correlation coefficients* c_{ij} *and there exist vectors* $\mathbf{w}, \mathbf{w}^* \in \{0,1\}^n$*, such that* $\mathbf{w}$ *linearly separates the list* $L = \langle\langle \mathbf{c}_1, w_1^*\rangle, \ldots, \langle \mathbf{c}_n, w_n^*\rangle\rangle$ *but* $\mathbf{w}^*$ *does not linearly separate* L*. Thus the list* L *is linearly separable but the target vector* $\mathbf{w}^*$ *cannot be learned by STDP.*

Proof: Consider the matrix and vectors

$$C = \begin{pmatrix} 1 & 0.25 & 0.1 & 0.5 & 0 \\ 0.25 & 1 & 0.1 & 0.5 & 0 \\ 0.1 & 0.1 & 1 & 0.05 & 0 \\ 0.5 & 0.5 & 0.05 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{w}^* = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

The list $L = \langle\langle \mathbf{c}_1, w_1^* \rangle, \ldots, \langle \mathbf{c}_n, w_n^* \rangle\rangle$ where $\mathbf{c}_i$ is the i-th row vector of C is not separated by $\mathbf{w}^*$, because $C\mathbf{w}^* = (1.35, 1.35, 1.2, 1.05, 1)^T$. However, $\mathbf{w}$ separates L because $C\mathbf{w} = (0.1, 0.1, 1, 0.05, 1)^T$.

In the following, we show how one can construct binned spiketrains $S_1, \ldots, S_5$ with rate r and correlation matrix C for arbitrarily small binsize ΔT. In the limit of $\Delta t \to 0$, this construction produces Poisson spike trains. The first spiketrain S_1 is constructed by assigning a spike to each bin with probability $r\Delta T$.

Let $X_i \in \{0, 1\}$ be a random variable which is 1 if S_i has a spike in a given bin, and 0 otherwise. In order to construct S_2, we compute the probability ϕ_{21} of placing a spike into a bin where S_1 has a spike. We do that with probability $\phi_{21} = P[X_2 = 1 | X_1 = 1]$. We need to make sure that $C_{21}^0(\Delta k) = c_{21}\delta_{\Delta k,0}/(r\Delta T)$, where $C_{21}^0(\Delta k)$ denotes the correlation beween two bins separated by Δk. We have

$$\begin{aligned} C_{ij}^0(\Delta k) &= \delta_{\Delta k,0}\left(\frac{P[X_1 = 1, X_2 = 1]}{r^2\Delta T^2} - 1\right) \\ &= \delta_{\Delta k,0}\left(\frac{P[X_1 = 1]P[X_2 = 1|X_1 = 1]}{r^2\Delta T^2} - 1\right) \\ &= \delta_{\Delta k,0}\left(\frac{r\Delta T\phi_{21}}{r^2\Delta T^2} - 1\right) \\ &= \delta_{\Delta k,0}\left(\frac{\phi_{21}}{r\Delta T} - 1\right) . \end{aligned}$$

Since, $C_{21}^0(\Delta k) = c_{21}\delta_{\Delta k,0}/(r\Delta T)$, we obtain $\phi_{21} = c_{21} + r\Delta T$. We then put spikes into bins where S_1 does not have a spike with probability Θ_2. Details on how to calculate Θ_1 are given below.

The third spike train can be constructed in the same way. We place a spike into a bin where *only* S_1 has a spike with probability $\phi_{31} = P[X_3 = 1 | X_1 = 1, X_2 = 0]$, and we place a spike into a bin where *only* S_2 has a spike with probability $\phi_{32} = P[X_3 = 1|X_1 = 0, X_2 = 1]$. These probabilities are given by

$$\phi_{31} = \phi_{32} = \frac{c_{31} + r\Delta T}{1 - \phi_{21}} \,. \tag{1}$$

Again, we put spikes into bins where non of the previous spike trains had a spike with probability Θ_3 (details given below).

Spike train S_4 needs special attention. The method we used above does not work because the sum of correlations to S_1, S_2, and S_3 are larger than 1. Hence, using the above method would lead to a rate larger than r. We construct S_4 by placing a spike into a bin where S_1 *and* S_2 has a spike with probability 1. This achieves some correlation between S_4 and S_1, S_2. We achieve the desired correlation by placing a spike into a bin where *only* S_1 has a spike with probability $\phi_{41} = P[X_4 = 1|X_1 = 1, X_2 = 0, X_3 = 0]$. We place a spike into a bin where *only* S_2 has a spike with probability $\phi_{42} = P[X_4 = 1|X_1 = 0, X_2 = 1, X_3 = 0]$. These probabilities are given by

$$\phi_{41} = \phi_{42} = \frac{c_{41} - c_{21}}{(1 - \phi_{21})(1 - \phi_{31})} \,. \tag{2}$$

The probability $\phi_{43} = P[X_4 = 1|X_1 = 0, X_2 = 0, X_3 = 1]$ is given by

$$\phi_{43} = \frac{(c_{43} + r\Delta T)(1 - r\Delta T)(1 - \phi_{21}))}{(1 - \phi_{21} - 2\phi_{31}(1 - \phi_{21}))(1 - r\Delta T(2 - \phi_{21}))} \,. \tag{3}$$

Again, we put spikes into bins where non of the previous spike trains have a spike with probability Θ_4 (details given below). For $\Delta T \to 0$, these probabilites converge to a value in the interval $(0, 1)$. The following limit values can be shown easily: $\lim_{\Delta T \to 0} \phi_{21} = c_{21} = 1/4$, $\lim_{\Delta T \to 0} \phi_{31} = 2/15$, $\lim_{\Delta T \to 0} \phi_{41} = 5/13$, and $\lim_{\Delta T \to 0} \phi_{43} = 1/12$.

The last spike train is uncorrelated with all other spike trains. We place a spike in any bin with probability $r\Delta T$.

Spike trains S_1 and S_5 have the correct rate. The other spike trains need to be adjusted such that for each bin, the probability of a spike is $r\Delta T$, i.e. $P[X_i = 1] = r\Delta T$. We achieve this by adjusting the probability that S_i spikes in a bin, given that no other spike train has a spike in this bin (except for S_5). We denote this probability by Θ_i.

In the following we show that such Θ_i exist. Denote with X_i' the random variables according the spike trains we constructed so far, i.e. without spikes given by Θ_i. We have $P[X_i = 1] = P[X_i' = 1] + P[\sum_{j<i} X_j = 0]\Theta_i$. With the use of the constraint $P[X_i' = 1] + P[\sum_{j<i} X_j = 0]\Theta_i = r\Delta T$, one can determine Θ_i as

$$\Theta_i = \frac{r\Delta T - P[X_i' = 1]}{P[\sum_{j<i} X_j = 0]}. \tag{4}$$

To proof that such probabilites Θ_i exist, we will show that $\Theta_i > 0$ for small enough ΔT, and that for $\Delta T \to 0$, we have $\Theta_i \to 0$. For S_2, we find

$$P[X_2' = 1] = P[X_1 = 1]\phi_{21} = r\Delta T\phi_{21} \; .$$

Since $P[X_1 = 0] = 1 - r\Delta T$, we get

$$\Theta_2 = \frac{r\Delta T(1 - \phi_{21})}{1 - r\Delta T} \; .$$

Since $P[X_1 = 1, X_2 = 0] = P[X_1 = 0, X_2 = 1]$, we find

$$\begin{aligned} P[X_3' = 1] &= P[X_1 = 1, X_2 = 0]\phi_{31} + P[X_1 = 0, X_2 = 1]\phi_{32} \\ &= 2\phi_{31}P[X_1 = 1, X_2 = 0] = 2r\Delta T\phi_{31}(1 - \phi_{21}) \; . \end{aligned}$$

Since, $P[X_1 = 0 X_2 = 0] = 1 - r\Delta T(1 - \phi_{21})$, we obtain

$$\Theta_3 = \frac{r\Delta T(1 - 2\phi_{31}(1 - \phi_{21}))}{1 - r\Delta T(1 - \phi_{21})} \; .$$

Obviously, Θ_2 and Θ_3 converge to zero. One can easily show that they Θ_i is also positive, by showing $\lim_{\Delta T \to 0} \frac{P[X_i'=1]}{r\Delta T} < 1$. We will not give a formula for Θ_4 explicitly, but concentrate on showing that $\lim_{\Delta T \to 0} \frac{P[X_4'=1]}{r\Delta T} < 1$. With the help of the equality $P[X_1 = 0|X_2 = 1] = P[X_2 = 0|X_1 = 1]$, we find

$$\begin{aligned}
P[X_4' = 1] &= P[X_1 = 1, X_2 = 1] + P[X_1 = 1, X_2 = 0, X_3 = 0]\phi_{41} \\
&\quad + P[X_1 = 0, X_2 = 1, X_3 = 0]\phi_{42} \\
&\quad + P[X_1 = 0, X_2 = 0, X_3 = 1]\phi_{43} \\
&< P[X_1 = 1]P[X_2 = 1|X_1 = 1] \\
&\quad + \phi_{41}P[X_1 = 1]P[X_2 = 0|X_1 = 1]P[X_3 = 0|X_1 = 1X_2 = 0] \\
&\quad + \phi_{42}P[X_2 = 1]P[X_1 = 0|X_2 = 1]P[X_3 = 0|X_1 = 0X_2 = 1] \\
&\quad + \phi_{43}P[X_3 = 1] \\
&= r\Delta T(\phi_{21} + 2\phi_{41}(1 - \phi_{21})(1 - \phi_{31}) + \phi_{43})\ .
\end{aligned}$$

It is easy to show that $\lim_{\Delta T \to 0} \frac{P[X_4'=1]}{r\Delta T} < \frac{5}{6}$. Hence, Θ_4 is positive for large enough ΔT (the probability in the denominator is positive). To show that Θ_4 converges to zero, we show that $P[X_1 = 0, X_2 = 0, X_3 = 0]$ converges to 1. Note that $P[X_1 = 0, X_2 = 0, X_3 = 0] = P[X_1 = 0, X_2 = 0]P[X_3 = 0|X_1 = 0, X_2 = 0] = 1 - r\Delta T(1 - \phi_{21})(1 - \Theta_3)$. Since Θ_3 converges to zero, we have $\lim_{\Delta T \to 0} \Theta_4 = 0$.

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