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RECURSIVELY ENUMERABLE GENERIC SETS

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Abstract. We show that one can solve Post's Problem by constructing generic sets in the usual set theoretic framework applied to tiny universes. This method leads to a new class of recursively enumerable sets: r.e. generic sets. All r.e. generic sets are low and simple and therefore of Turing degree strictly between **0** and **0**′. Further they supply the first example of a class of low recursively enumerable sets which are automorphic in the lattice $\mathscr{E}$ of recursively enumerable sets with inclusion. We introduce the notion of a promptly simple set. This describes the essential feature of r.e. generic sets with respect to automorphism constructions.

It is obvious that constructions of generic sets in set theory and certain constructions of recursively enumerable (r.e) sets have something in common. It is typical for the construction of a generic set U that a sentence ϕ about U is finally made true (i.e. $M[U] \models \phi$) if one has unboundedly often the chance to make it true during the construction of U (i.e. $\forall p\, \exists q \leq p\, (q \Vdash \phi)$).

On the other hand consider the construction of a simple r.e. set G in recursion theory (G is simple if it has an infinite complement but $G \cap W \neq \varnothing$ for every infinite r.e. set W). The strategy is to make the sentence $\exists x\, (x \in W \wedge x \in G)$ true for every r.e. set W where one has infinitely often the chance to do so without filling up the complement of G. Thus one succeeds at least for every infinite r.e. set W.

Similarly one constructs an r.e. set G such that the components $(G)_0$, $(G)_1$ ($(G)_i: = \{x \mid \langle i, x\rangle \in G\}$) are of incomparable Turing degree. The sentences $\exists x((G)_0(x) \neq \{e\}^{(G)_1}(x))$ and $\exists x\, ((G)_1(x) \neq \{e\}^{(G)_0}(x))$ are finally made true for every e where one has infinitely often the chance to make them true without filling up the complement of $(G)_0$, $(G)_1$. Then we cannot have $(G)_0 = \{e\}^{(G)_1}$ because such an e would offer infinitely often the chance to make the first sentence true. Therefore $(G)_0$, $(G)_1$ are incomparable.

Observe that one can describe the desire to make the complement of G infinite as well as follows: for every n one tries to satisfy the sentence $\exists x\, (x > n \wedge x \notin G)$ if one has infinitely often the chance (which one has here for every n).

In this paper we study constructions of r.e. sets which resemble even more the constructions of generic sets in set theory. In particular we do not restrict our

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attention to a few special sentences about the constructed r.e. set. Instead we consider all sentences of possible recursion theoretic interest.

In §1 we define r.e.-generic sets and prove some basic properties.

In §2 we introduce promptly simple sets.

In §3 we combine Soare's automorphism constructions ([4] and [6]) in order to prove that all promptly simple sets with semilow complement are automorphic.

Further information about promptly simple sets will be contained in a forthcoming joint paper of R. Shore, M. Stob and the author.

§1. **R.e. generic sets.** We use the usual framework for forcing in set theory (see e.g. Jech's book [1]). For elements p, q of a partially ordered set $(\mathscr{P}, \leq)$ one says that p is stronger than q if $p < q$. A set $D \subseteq \mathscr{P}$ is dense if $\forall p \in \mathscr{P}\ \exists q \in D\ (q \leq p)$. For any collection $\mathscr{D}$ of dense sets in $\mathscr{P}$ one says that a set $U \subseteq \mathscr{P}$ is $\mathscr{D}$-generic if:

1. U is a filter on $\mathscr{P}$ (i.e. $U \neq \emptyset$, $\forall pq \in \mathscr{P}\ (p \leq q \wedge p \in U \rightarrow q \in U)$ and $\forall pq \in U\ \exists r \in U\ (r \leq p \wedge r \leq q)$);
2. $U \cap D \neq \emptyset$ for every $D \in \mathscr{D}$.

In our forcing approach the inner model M consists of HF (the collection of hereditarily finite sets) together with all primitive recursive subsets of HF. We define a partial ordering $(\mathscr{P}, \leq) \in M$ and we try to construct generic paths through $\mathscr{P}$ which meet as many dense sets $D \in M$ as possible. The outer model N—in which the construction of generic sets takes place—is HF together with all recursive subsets of HF. One could take instead various other models M and N (e.g. models of certain weak theories). As in set theory the essential point is that M is countable inside N: There is some $S \in N$, $S \subseteq \omega \times HF$, such that $M = \{\{x \in HF \mid (i, x) \in S\} \mid i \in \omega\}$.

The main difference to the usual forcing constructions in set theory is due to the fact that the outer universe N is very small. There are many sets $D \subseteq \mathscr{P}$, $D \in M$ which are dense but which cannot be recognized as being dense from inside N because there is no sufficiently effective procedure to compute from a given $p \in \mathscr{P}$ some $q \in D$ with $q \leq p$. But if we construct a generic set inside N we can only hope to meet all those dense sets $D \in M$ where the density of D can be seen inside N. Therefore we miss a lot of dense sets, in particular most of the sets $D_\phi = \{p \mid p \Vdash \phi \vee p \Vdash \neg \phi\}$ (with a forcing relation $p \Vdash \phi$ defined as usual). This implies that there is no adequate forcing relation for generic sets in N (at least not for the considered languages).

On the other hand it is advantageous to work in such a small outer universe N. Every element of N is easily definable. Therefore we can eliminate U from a sentence which describes the generic extension $M[U]$ by plugging in the definition of U (in this way we show in Lemma 7 that all r.e. generic sets are low).

Our partial order $\mathscr{P} \in M$ has the property that every path through $\mathscr{P}$ is a function which enumerates a certain set. We will define a class $\mathscr{D} \subseteq M$ of dense subsets of $\mathscr{P}$ such that $\mathscr{D}$-generic sets U exist in N. We call a set G r.e. generic if it is enumerated by such a recursive $\mathscr{D}$-generic path U through $\mathscr{P}$.

We are modest enough not to expect that we get information about all aspects of the generic extension $M[U]$. We are mainly interested in the properties of the structure $\langle HF, \in \restriction HF \times HF, G\rangle$, where G is the set enumerated by U.

We consider a language $(S_n)_{n\in\omega}$ which is adequate for the description of this structure. The partial order $\mathscr{P}$ is defined in such a way that we get some information about the final truth of formulas S_n by studying the corresponding path through $\mathscr{P}$. This method works particularly well if S_n is a Σ_1^G formula.

For the following fix an enumeration $(S_n)_{n\in\omega}$ of all sentences in first order logic with a two-place predicate $\in$ and a one-place predicate G and with elements of HF as parameters. We define a hierarchy for HF as follows:

$$V_0 := \emptyset, \quad V_{n+1} := \{x \mid x \subseteq V_n\}, \quad V_\omega := \bigcup_{n\in\omega} V_n.$$

Then $V_\omega = HF$ and, for every $n \in \omega$, V_n is transitive, $V_n \cap \omega = n$ and $V_n \subsetneq V_{n+1}$.

Relative to any given sequence $\langle G_0, \ldots, G_k\rangle$ of sets $G_i \subseteq i$, we define, for $0 \le i \le j \le k$ by recursion on i,

$$t^j(i) = \begin{cases} \mu n \in \{t^j(i-1)+1, \ldots, j\} \\ \quad (\langle V_n, \in \restriction V_n \times V_n, G_n\rangle \models S_i), \\ \qquad\qquad\qquad \text{if such an } n \text{ exists,} \\ \min\{t^j(i-1)+1, j\}, \quad \text{otherwise.} \end{cases}$$

We set $t^j(-1) := -1$.

This is our partially ordered set:

$$\mathscr{P} := \{\langle G_0, \ldots, G_k\rangle \mid k \ge 0 \wedge G_0 \subseteq \cdots \subseteq G_k \wedge \forall i \in \{0, \ldots, k\}\,(G_i \subseteq i) \wedge \forall ij \in \{0, \ldots, k\}\,(i \le j \to G_j \cap t^j(i) = G_{t^j(i)})\} \cup \{\langle\,\rangle\}.$$

For $p, q \in \mathscr{P}$ we define

$$p \le q :\Leftrightarrow q \text{ is an initial segment of } p.$$

We define $\mathscr{D}$ as the class of primitive recursive sets $D \subseteq \mathscr{P}$ for which a primitive recursive function g_D with the following property exists: if $\langle G_0, G_1, \ldots\rangle$ is an infinite recursive sequence with all initial segments in $\mathscr{P}$ then

$$\forall k \exists n \ge k(g_D(\langle G_0, \ldots, G_n\rangle) \in D \wedge g_D(\langle G_0, \ldots, G_n\rangle) < \langle G_0, \ldots, G_n\rangle).$$

Notation. In the future we write mostly $\langle V_n, G_n\rangle$ instead of $\langle V_n, \in \restriction V_n \times V_n, G_n\rangle$, etc.

LEMMA 1. (a) *If $\langle G_0, \ldots, G_k\rangle \in \mathscr{P}$ and $j \le k$ then the function t^j depends only on $\langle G_0, \ldots, G_j\rangle$ and there is some $i_0 \le j$ such that*

$$0 \le t^j(0) < t^j(1) < \cdots < t^j(i_0) = \cdots = t^j(j) = j.$$

Further $t^{j+1}(i) = t^j(i)$ or $t^{j+1}(i) = j + 1$ for

$$i \le j < k \text{ and } t^j(i) \le t^{j'}(i) \text{ for } i \le j \le j' \le k.$$

(b) *If $\langle G_0, \ldots, G_k\rangle \in \mathscr{P}$ then every initial segment of $\langle G_0, \ldots, G_k\rangle$ and $\langle G_0, \ldots, G_{k-1}, G_k, G_k\rangle$ are as well elements of $\mathscr{P}$.*

(c) *If* $\langle G_0 \rangle > \langle G_0, G_1 \rangle > \ldots$ *is an infinite path in* $\mathscr{P}$ *then*

$$t(i) := \lim_{j \to \omega} t^j(i) \text{ exists for every } i \geq -1.$$

Further if $i \geq -1$ *and* $t^j(i) = t(i)$ *for all* $j \geq j_0$ *then*

$$\text{either } \forall k > t(i)(\langle V_k, G_k \rangle \models \neg S_{i+1}) \text{ and}$$
$$t^j(i+1) = t(i) + 1 \text{ for all } j > j_0$$
$$\text{or there is some minimal } k_0 > t(i) \text{ with}$$
$$\langle V_{k_0}, G_{k_0} \rangle \models S_{i+1} \text{ and } t^j(i+1) = k_0 \text{ for all } j \geq \max\{j_0, k_0\}.$$

Thus we have for every $i \geq 0$,

$$\langle V_{t(i)}, G_{t(i)} \rangle \models S_i$$

or

$$\langle V_k, G_k \rangle \models \neg S_i \quad \text{for all } k \geq t(i).$$

If S_i *is a* Σ_1^G *sentence (with predicates* $\in$ *and* G*) we have, in addition,*

$$\langle V_{t(i)}, G_{t(i)} \rangle \models S_i \Rightarrow \forall k \geq t(i)(\langle V_k, G_k \rangle \models S_i)$$
$$\wedge \langle V_\omega, \bigcup_{k \in \omega} G_k \rangle \models S_i.$$

This implies for all Σ_1^G *sentences* S_i,

$$\langle V_\omega, \bigcup_{k \in \omega} G_k \rangle \models S_i \Leftrightarrow \forall n \exists k \geq n(\langle V_k, G_k \rangle \models S_i).$$

Proof. Straightforward verification.

Lemma 2. *Assume that* U *is* $\mathscr{D}$*-generic. Then there is an infinite sequence* $\langle G_0, G_1, \ldots \rangle$ *such that* U *consists exactly of the finite initial segments of this sequence.*

Proof. For $p, q \in U$ there is some $r \in U$ such that $r \leq p$ and $r \leq q$. Therefore p is an initial segment of q or q is an initial segment of p. It remains only to show that for every $k \in \omega$ there is some $p \in U$ of length $\geq k$. But this follows from U $\mathscr{D}$-generic since

$$D := \{p \in \mathscr{P} \mid p \text{ has length } \geq k\} \in \mathscr{D}$$

(define $g_D(\langle G_0, \ldots, G_n \rangle) := \langle G_0, \ldots, G_{n-1}, G_n, G_n \rangle$).

Intuitively an element $\langle G_0, \ldots, G_k \rangle$ of $\mathscr{P}$ should be considered as an initial segment—the first $k+1$ steps—of a construction of an r.e. set $G = \bigcup_{n \in \omega} G_n$. G_n corresponds to the set of elements which are enumerated in G by step n of the construction. The condition $G_j \cap t^j(i) = G_{t^j(i)}$ demands that small numbers are enumerated at a later step only if one has a good reason. Essentially one tries to freeze for every sentence S_i the first structure $\langle V_n, G_n \rangle$ where S_i holds, i.e. we would like to get $G \cap n = G_n$. Nevertheless at a later step $m > n$ one may enumerate a new element less than n in G (and thus make $G_m \cap n \neq G_n$) if one can satisfy in $\langle V_m, G_m \rangle$ some sentence $S_{i'}$ with $i' < i$, which did not hold in any previous structure. Assume for example $i' = 0$. Then $t^m(0) = m$ and thus $t^m(l) = m$ for all $l \leq m$. In this case the condition $G_m \cap t^m(l) = G_{t^m(l)}$ for $l \leq m$ does not put any restriction on G_m.

Observe that one could write the "forcing conditions" for our approach as well as finite partial functions $f: \omega \to \omega$. $\mathrm{Rg}\, f$ plays then the role of G_n. Forcing conditions of this type are more common in set theory. Sequences $\langle G_0, \ldots, G_k \rangle$ have the advantage that we may enumerate at every step several new elements or none. This is convenient for constructions in recursion theory.

The definition of $\mathscr{D}$ is motivated by the desire to characterize the largest collection of dense sets in the inner model M such that $\mathscr{D}$-generic sets exist in N. One can easily see that it is not possible to meet all $D \in M$ which are effectively dense, i.e. where a total recursive function f exists such that for every $p \in \mathscr{P}$ we have $f(p) \leq p$ and $f(p) \in D$. On the other hand if we require that f is primitive recursive we can meet all these sets in N but the corresponding generic sets do not have interesting properties. The previously defined collection $\mathscr{D}$ lies between both extremes.

Let us consider how a set theorist—working in $V \supsetneq N$—would construct a generic path $p_0 \geq p_1 \geq \cdots$ through $\mathscr{P}$. He lists all sets $D \in M$ with $D \subseteq \mathscr{P}$: D_0, $D_1, \ldots$. Then in order to define p_{n+1} he checks whether there is some $p \leq p_n$ with $p \in D_{n+1}$. If yes, he defines $p_{n+1} := p$, otherwise he takes $p_{n+1} := p_n$. This decision is ineffective and cannot be made inside N. But we can do the following. We apply the first n primitive recursive functions $f_0, \ldots, f_n$ to p_n. Then we check whether one of the elements $f_0(p_n), \ldots, f_n(p_n)$ is an element of D_{n+1} and stronger than p_n. Further our construction becomes more powerful if we do not insist that we meet D_{n+1} at step $n + 1$ or never. Thus we check whether one of the elements $f_0(p_n)$, $\ldots, f_n(p_n)$ is in one of the sets $D_0, \ldots, D_{n+1}$ which we did not yet meet before. In this way one can meet all $D \in \mathscr{D}$ as we verify by the following construction.

THEOREM 3. *There exist recursive $\mathscr{D}$-generic sets.*

PROOF. We construct an infinite recursive path $p_0 > p_1 > \cdots$ in $\mathscr{P}$. At step n of the construction we define p_n. Fix for the construction a recursive enumeration $(f_j)_{j \in \omega}$ of all primitive recursive functions. Set $p_0 := \langle\ \rangle$.

Step $n + 1$. If there is no $\langle i, j \rangle \leq n$ such that $f_j(p) = 1$ for all $p \geq p_n$ and $f_i(p_n) \in \{p \in \mathscr{P} \mid p < p_n \wedge f_j(p) = 0\}$ we set $p_{n+1} := \langle G_0, \ldots, G_{m-1}, G_m, G_m \rangle$, where $\langle G_0, \ldots, G_m \rangle := p_n$. (*Convention*: the characteristic function of a set has value 0 for arguments in the sets, value 1 otherwise.) Otherwise we call the least such pair $\langle i_1, j_1 \rangle$ and set $q_1 := p_n$. Once we have defined $\langle i_e, j_e \rangle, q_e$ for some $e \geq 1$ we check whether there is some $\langle i, j \rangle < \langle i_e, j_e \rangle$ and some $q \in \mathscr{P}$ with $q_e > q > f_{i_e}(q_e)$ such that $f_j(p) = 1$ for all $p \geq q$ and $f_i(q) \in \{p \in \mathscr{P} \mid p < q \wedge f_j(p) = 0\}$. If it exists, we call the least such pair $\langle i_{e+1}, j_{e+1} \rangle$ and the minimal q for this pair q_{e+1}. Obviously there is a maximal $k \geq 1$ such that $\langle i_k, j_k \rangle$, q_k are defined. We set $p_{n+1} := f_{i_k}(q_k)$ for this maximal k.

We show that $U := \{p \in \mathscr{P} \mid \exists n(p \geq p_n)\}$ is $\mathscr{D}$-generic. Let $D \in \mathscr{D}$ be given with characteristic function f_{j_0}. Let f_{i_0} be the function g_D associated with D. Observe that for every $e \in \omega$ there is at most one step $n + 1$ of the construction where we have $j_k = e$ for the maximal k. Therefore there is some $n_0 \geq \langle i_0, j_0 \rangle$ such that at every step $n + 1$ with $n \geq n_0$ we have $\langle i_k, j_k \rangle \geq \langle i_0, j_0 \rangle$ for the maximal k. Since

$D \in \mathscr{D}$ there is some $n > n_0$ and some $q \in \mathscr{P}$ such that $p_n \geq q > p_{n+1}$ and $f_{i_0}(q) \in \{p \in \mathscr{P} | p < q \wedge f_{j_0}(p) = 0\}$. Assume that for all $p \geq q$, $f_{j_0}(p) = 1$ (otherwise we are finished). Then one defines a sequence $p_n = q_1 > \cdots > q_k > f_{i_k}(q_k) = p_{n+1}$ at step $n + 1$. If $q = q_e$ for some $e \leq k$ then $\langle i_e, j_e \rangle \leq \langle i_0, j_0 \rangle$. Otherwise $q_k > q$ (then we have $\langle i_k, j_k \rangle \leq \langle i_0, j_0 \rangle$) or $q_e > q > q_{e+1}$ for some $e < k$ which implies $\langle i_{e+1}, j_{e+1} \rangle \leq \langle i_0, j_0 \rangle$. In any case we get $\langle i_k, j_k \rangle \leq \langle i_0, j_0 \rangle$ and thus $\langle i_k, j_k \rangle = \langle i_0, j_0 \rangle$ by the choice of n_0. Therefore $f_{j_0}(p_{n+1}) = 0$.

Notice that the slight complication in the construction is due to the fact that in general we cannot expect that the element q above—which is given by the definition of $D \in \mathscr{D}$—is of the form p_n for some n.

DEFINITION 4. A set G is called r.e. generic if $G = \bigcup_{i \in \omega} G_i$ for a sequence $\langle G_0, G_1, \ldots \rangle$ such that the finite initial segments of this sequence form a recursive $\mathscr{D}$-generic set. We say then that $\langle G_0, G_1, \ldots \rangle$ represents G.

In the following lemmata we prove some standard properties of r.e. generic sets G. Most proofs have the same pattern. We want to show that a certain Σ_1^G sentence S_i holds in $\langle V_\omega, G \rangle$. By Lemma 1(c) it is enough to show for the representing $\mathscr{D}$-generic sequence $\langle G_0, G_1, \ldots \rangle$ that $\forall n \exists m \geq n(\langle V_m, G_m \rangle \models S_i)$. Thus we just have to show that for every $n_0 \in \omega$,

$$(*) \qquad D := \{\langle H_0, \ldots, H_k \rangle \in \mathscr{P} | \exists m\, (n_0 \leq m \leq k \wedge \langle V_m, H_m \rangle \models S_i)\}$$

is an element of $\mathscr{D}$.

LEMMA 5. *If G is r.e. generic and M is an infinite primitive recursive set then $\bar{G} \cap M$ is infinite.*

PROOF. Take $S_i :\equiv \exists x(\phi(x) \wedge k_0 < x \wedge x \notin G)$ where ϕ is a primitive recursive definition of M over V_ω and k_0 is a given natural number. Then for any n_0, $(*)$ is an element of $\mathscr{D}$ with associated function

$$g_D(\langle H_0, \ldots, H_k \rangle) = \langle H_0, \ldots, H_{k-1}, H_k, \ldots, H_k \rangle$$

(we add $h(k)$ many copies of H_k at the end where h is a primitive recursive function such that for every $k \in M$, $V_{h(k)} \models \phi(k)$).

LEMMA 6. *Assume that G is r.e. generic, represented by $\langle G_0, G_1, \ldots \rangle$.*

(a) *If W is an infinite r.e. set then $G \cap W \neq \emptyset$.*

(b) *There is a total recursive function h such that for every infinite r.e. set W and every Σ_1 formula ϕ which defines W over V_ω we have*

$$\exists x\, (x \in W \wedge x \in G_{h(\mu n(\langle V_n \rangle \models \phi(x)))}).$$

PROOF. A direct proof of (a) is very easy: Define $S_i := \exists x\, (x \in W \wedge x \in G)$ and proceed as before.

Claim (b) is stronger than (a) (it shows that G is promptly simple; see §2). In order to show (b) we consider a sentence S_i which says "there is an x which was just before enumerated in W and which is already an element of G". Then it is obvious that for W infinite one has at infinitely many stages n the opportunity to make S_i true in $\langle V_n, G_n \rangle$. The considered formula S_i cannot be written as a Σ_1^G formula because we need a universal quantifier in order to express "just before" inside V_n. In order to give an exact definition of S_i take some formula ψ such that for every $n \in \omega$ and $u, v \in V_n$,

$$\langle V_n \rangle \models \psi(u, v) \Rightarrow u \in n \wedge v = V_u$$

and such that the primitive recursive function g defined by $g(k) = \mu n(\langle V_n \rangle \models \psi(k, V_k))$ is strictly increasing. Using ψ one can easily define a formula χ such that for every $n \in \omega$ and $x \in V_n$,

$$\langle V_n \rangle \models \chi(x) \Leftrightarrow \exists k < n(g(k) \leq n < g(k+1) \wedge x \in W \wedge k = \mu n(\langle V_n \rangle \models \phi(x)))$$

where ϕ is a Σ_1 definition of W over V_ω. Intuitively $\langle V_n \rangle \models \chi(x)$ says that x was just before enumerated into W, i.e. for $k_0 := \mu k(\langle V_k \rangle \models \phi(x))$ one can already build V_{k_0} inside V_n but one cannot yet build V_{k_0+1} inside V_n (or equivalently $g(k_0) \leq n < g(k_0 + 1)$). One needs here a universal quantifier in χ in order to express over V_n that V_{k_0+1} cannot be built inside V_n, where $k_0 \in V_n$.

We define $S_i :\equiv \exists x(\chi(x) \wedge x \in G)$. The desired recursive function h is defined by $h(k) := g(k + 1)$.

Assume then that the r.e. set W which is defined by the Σ_1 formula ϕ over V_ω is infinite. We have to show that there is some m such that $\langle V_m, G_m \rangle \models S_i$. Obviously it is enough to show that

$$D := \{\langle H_0, \ldots, H_k \rangle \in \mathscr{P} \mid \exists m \leq k(\langle V_m, H_m \rangle \models S_i)\}$$

is an element of $\mathscr{D}$. Define a primitive recursive function g_D by

$$g_D(\langle H_0, \ldots, H_k \rangle) = \begin{cases} \langle H_0, \ldots, H_k, H_k \cup \{x\} \rangle, & \text{where } x < k \text{ is minimal such that this is in } D, \text{ if such an } x \text{ exists;} \\ \langle H_0, \ldots, H_{k-1}, H_k, H_k \rangle, & \text{otherwise.} \end{cases}$$

Consider any given recursive path $\langle H_0, H_1, \ldots \rangle$ through $\mathscr{P}$. Define $t(i)$ for this sequence as in Lemma 1(c). If $\langle V_{t(i)}, H_{t(i)} \rangle \models S_i$ then there is trivially some k such that $g_D(\langle H_0, \ldots, H_k \rangle) \in D$. Otherwise we have to wait until $k = g(\mu n(\langle V_n \rangle \models \phi(x))) - 1$ for some $x \in W$ with $x \geq t(i)$. At this point g_D is defined according to the first clause of the definition and provides an element of D.

LEMMA 7. *Every r.e. generic set G is low (i.e. $G' = 0'$).*

PROOF. Let the recursive sequence $\langle G_0, G_1, \ldots \rangle$ represent G. By Lemma 1(c) we have for every Σ_1^G formula ϕ:

$$\langle V_\omega, G \rangle \models \neg \phi \Leftrightarrow \exists n_0 \forall n \geq n_0(\langle V_n, G_n \rangle \models \neg \phi).$$

Since the sequence $\langle G_0, G_1, \ldots \rangle$ is recursive we can write the right side as a Σ_2 formula.

LEMMA 8. *Assume G is r.e. generic and M_0, M_1 are disjoint infinite primitive recursive sets. Then the r.e. sets $G \cap M_0$, $G \cap M_1$ have incomparable Turing degree.*

PROOF. Assume for a contradiction that W is an r.e. set such that for every $x \in \omega$,

$$x \in \overline{G \cap M_0} \Leftrightarrow \exists \text{ finite } H(\langle x, H \rangle \in W \wedge H \subseteq \overline{G \cap M_1}).$$

Take a Σ_1^G formula S_i which asserts that there is a counterexample:

$$S_i :\equiv \exists x\,(x \in G \cap M_0 \wedge \exists \text{ finite } H$$
$$(\langle x, H\rangle \in W \wedge H \subseteq \overline{G \cap M_1})).$$

By Lemma 1(c) we have reached a contradiction if we can show that $\forall n\ \exists m \geq n$ $(\langle V_m, G_m\rangle \models S_i)$, where $\langle G_0, G_1, \ldots\rangle$ represents G. Obviously along the path $\langle G_0, G_1, \ldots\rangle$ we have unboundedly often the chance to make S_i true. But in order to define a corresponding dense set $D \in \mathscr{D}$ we have to make sure that D can be reached from every recursive path $\langle H_0, H_1, \ldots\rangle$ through $\mathscr{P}$.

In order to define D we consider $t(i-1)$ for $\langle G_0, G_1, \ldots\rangle$ as in Lemma 1(c). By Lemma 5 there is some $x > t(i-1)$ with $x \in \bar{G} \cap M_0$. Consider the corresponding computation from $G \cap M_1$ via W. Fix some finite H such that $\langle x, H\rangle \in W$ and $H \subseteq \overline{G \cap M_1}$. Split H into F_1, F_2 such that $F_1 \subseteq \bar{G} \cap M_1$ and $F_2 \subseteq \bar{M}_1$. Take n_0 large enough such that

$$\langle V_{n_0}\rangle \models [x \in M_0 \wedge \langle x, H\rangle \in W \wedge F_2 \subseteq \bar{M}_1].$$

Define

$D := \{\langle H_0, \ldots, H_{n+1}\rangle \in \mathscr{P} \mid n > n_0 \wedge$

(1) $\quad (\exists\, k \leq n\, (\langle V_k, H_k\rangle \models S_i) \vee$

(2) $\quad \neg(F_1 \cap H_n = \phi \wedge x \notin H_n \wedge t^{\langle H_0, \ldots, H_n\rangle}(i-1) < x) \vee$

(3) $\quad (F_1 \cap H_n = \phi \wedge x \notin H_n \wedge t^{\langle H_0, \ldots, H_n\rangle}(i-1) < x \wedge x \in H_{n+1}))\}$.

Define

$$g_D(\langle H_0, \ldots, H_n\rangle) = \begin{cases} \langle H_0, \ldots, H_{n-1}, H_n, H_n\rangle, & \text{if (1) or (2)} \\ \quad \text{holds for } \langle H_0, \ldots, H_n\rangle \text{ or if } n \leq n_0, & \\ \langle H_0, \ldots, H_n, H_n \cup \{x\}\rangle, & \text{otherwise.} \end{cases}$$

Consider any recursive path $\langle H_0, H_1, \ldots\rangle$ through $\mathscr{P}$. If $\langle H_0, \ldots, H_n\rangle$ satisfies (1) or (2) then $g_D(\langle H_0, \ldots, H_n\rangle)$ is defined according to the first clause and is obviously in D for $n > n_0$. If $g_D(\langle H_0, \ldots, H_n\rangle)$ is defined according to the second clause, then $\langle H_0, \ldots, H_n, H_n \cup \{x\}\rangle$ satisfies (3). An easy calculation shows that this value is then an element of $\mathscr{P}$ and therefore of D.

We have thus shown that $D \in \mathscr{D}$. Therefore there is some n such that $\langle G_0, \ldots, G_{n+1}\rangle \in D$. This can only happen because of (1) and so there is some $k \leq n$ such that $\langle V_k, G_k\rangle \models S_i$. In order to show that there are arbitrary large k with $\langle V_k, G_k\rangle \models S_i$, consider for a given k_0 instead of S_i a Σ_1^G formula S_j such that

$$\langle V_k, G_k\rangle \models S_j \Leftrightarrow k > k_0 \wedge \langle V_k, G_k\rangle \models S_i.$$

One shows analogously that $G \cap M_1$ is not Turing computable from $G \cap M_0$.

THEOREM 9. *Assume G is r.e. generic. Then G is low and simple and thus of Turing degree strictly between* **0** *and* **0**′. *Further the components $(G)_i := \{x \mid \langle i, x\rangle \in G\}$ are sets of independent Turing degrees: for any finite set $J \subseteq \omega$ and any $i \notin J$ we have $(G)_i \not\leq_T \bigvee_{j \in J} (G)_j$.*

PROOF. Use Lemmas 6–8.

§2. Promptly simple sets. Fix a simultaneous recursive enumeration of all the r.e. sets $(W_e)_{e\in\omega}$. Thus $\{\langle x, e\rangle \mid x \in W_{e,n}\}$ is finite for every n.

DEFINITION 10. A set $A \subseteq \omega$ is promptly simple if $\bar{A}$ is infinite and if there is an index i and a total recursive function f such that $W_i = A$ and for every $e \in \omega$,

$$W_e \text{ infinite} \Rightarrow \exists x(x \in W_e \wedge x \in W_{i, f(\mu n(x \in W_{e,n}))}).$$

The name is motivated by the fact that in practical cases one can usually choose $f(n) = n + 17$. The preceding dynamical definition is often useful in constructions. The following lemma shows that one can define it as well without reference to enumerations. We note that the notion of a promptly simple set is recursively invariant in the sense of Rogers [3].

LEMMA 11. *The following are equivalent for any r.e. set* W_i:

(a) W_i *is promptly simple.*

(b) $\bar{W}_i$ *is infinite and there is a total recursive nondecreasing function* f *such that for every* $e \in \omega$,

$$W_e \textit{ infinite} \Rightarrow \{x \mid x \in W_e \wedge x \in W_{i, f(\mu n(x \in W_{e,n}))}\} \textit{ infinite.}$$

(c) $\bar{W}_i$ *is infinite and there is a total recursive function* g *such that for every* $e \in \omega$,

$$W_{g(e)} \subseteq W_e \quad \textit{and} \quad W_{g(e)} \cap \bar{W}_i = W_e \cap \bar{W}_i$$

and W_e *infinite* $\Rightarrow W_e - W_{g(e)} \neq \emptyset$.

(d) *Same as in* (c) *with* "$W_e - W_{g(e)}$ *infinite*" *instead of* "$W_e - W_{g(e)} \neq \emptyset$".

PROOF. (a) $\Rightarrow$ (b): Assume $W_i = W_j$ and for all e,

$$W_e \text{ infinite} \Rightarrow \exists x(x \in W_e \wedge x \in W_{j, h(\mu n(x \in W_{e,n}))})$$

with a total recursive function h (h nondecreasing w.l.o.g.). Fix total recursive functions p and g_1 such that for every $e, k, n \in \omega$, $W_{p(e,k)} = W_e - k$ and

$$g_1(n) = \mu m(\forall x \in k((x \in W_{e,n} \wedge x \geq k) \rightarrow x \in W_{p(e,k),m})).$$

Define $g_2(n) := \mu m(W_{j,n} \subseteq W_{i,m})$. Then the recursive function f defined by

$$f(n) := \max\{g_2(h(g_1(m))) \mid m \leq n\}$$

has all the properties which are required in (b). If W_e is infinite then there is some $x \in W_e$ with $x \geq k$ and $x \in W_{j, h(\mu n(x \in W_{p(e,k),n}))}$. This implies $x \in W_{j, h(g_1(\mu n(x \in W_{e,n})))}$ and therefore $x \in W_{i, g_2(h(g_1(\mu n(x \in W_{e,n}))))}$.

(b) $\Rightarrow$ (d): Define $W_{g(e)} := \{x \in W_e \mid x \notin W_{i, f(\mu n(x \in W_{e,n}))}\}$.

Clearly (d) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a): Define

$$f(n) := \mu m \forall x e\, (x \in W_{e,n} \rightarrow (x \in W_{i,m} \vee x \in W_{g(e),m})).$$

Then if W_e is infinite there is some $x \in W_i$ with $x \in W_e \wedge x \notin W_{g(e)}$. We have for this x, $x \in W_{i, f(\mu n(x \in W_{e,n}))}$.

Lerman and Soare [2] have shown that not all simple sets of low degree are automorphic. They introduced the following notion.

DEFINITION 12. A is called d-simple if $\bar{A}$ is infinite and for all r.e. W there exists an r.e. $V \subseteq W$ such that $V \cap \bar{A} = W \cap \bar{A}$ and $(S - V) \cap A \neq \emptyset$ for every r.e. set S with $S - W$ infinite.

A is called uniformly d-simple if an index for V can be found effectively from one for W.

According to Lerman and Soare [2] there are low simple sets A and B such that A is d-simple but B is not.

THEOREM 13. (a) *A r.e. generic $\Rightarrow$ A promptly simple.*

(b) *A promptly simple $\Rightarrow$ A uniformly d-simple.*

PROOF. (a) Let G be r.e. generic with representing sequence $\langle G_0, G_1, \ldots \rangle$. Consider the following enumeration of the r.e. sets:

$$x \in W_{e,n} :\Leftrightarrow \langle V_n \rangle \models \phi_e(x),$$

where ϕ_e is the eth Σ_1 formula. Fix some index i such that $W_i = G$ and define a recursive g by

$$g(n) := \mu m(G_n \subseteq W_{i,m}).$$

Let h be the recursive function from Lemma 6(b). Define $f := g \circ h$.

(b) Assume W_i is promptly simple with witness f. Fix a total recursive function q such that for every $j, e \in \omega$,

$$W_{q(j,e)} = \{x \mid \exists n(x \in W_{j,n} \wedge x \notin W_{e,n})\}.$$

Define for $e \in \omega$ the r.e. set $Y_e \subseteq W_e$ by

$$Y_e := \{x \mid \exists n(x \in W_{e,n+1} - W_{e,n} \wedge \forall j < n \\ (x \in W_{j,n} \rightarrow x \notin W_{i,f(\mu m(x \in W_{q(j,e),m}))}))\}$$

(this definition of Y_e is due to Michael Stob and is more elegant than our original one). Obviously an r.e. index for Y_e can be computed from e. Further $Y_e \cap \bar{W}_i = W_e \cap \bar{W}_i$. Finally if $W_j - W_e$ is infinite then $W_{q(j,e)}$ is infinite and therefore there is some $x_0 \in W_{q(j,e)}$ with $x_0 \in W_{i,f(\mu m(x_0 \in W_{q(j,e),m}))}$ (since W_i is promptly simple). Obviously $x_0 \in W_j$. If $x_0 \notin W_e$ we are finished. Otherwise consider the n such that $x_0 \in W_{e,n+1} - W_{e,n}$. Since $x_0 \in W_{q(j,e)}$ we have $x_0 \in W_{j,n}$ and therefore $j < n$. In this case x_0 is excluded from Y_e by the second clause in the definition of Y_e. Thus $x_0 \in (W_j - Y_e) \cap W_i$.

COROLLARY 14. *Not every low r.e. degree contains an r.e. generic set. In fact there is an r.e. set A of nonzero degree such that no r.e. generic set G with $G \leq_T A$ exists.*

PROOF. Lerman and Soare [2] have shown the corresponding property for "d-simple" instead of "r.e. generic".

REMARK 15. An r.e. generic set G contains in fact many promptly simple sets in various degrees. All the sets $(G)_i$ in Theorem 9 are promptly simple.

§3. **Automorphisms.** We consider the lattice $\mathscr{E}$ of r.e. sets with inclusion. According to Soare [5] a set B is called semilow if the set $\{e \mid W_e \cap B \neq \emptyset\}$ is recursive in $\mathbf{0}'$. If A is low then $\bar{A}$ is semilow. But in addition every r.e. degree contains an r.e. set with semilow complement (see [5]). We show here that all promptly simple sets with semilow complement are automorphic.

Promptly simple sets A supply some sort of oracle in order to decide "$x \in A$?" for some number x which was just enumerated in W_e. The oracle says "yes" if $x \in A_{f(\mu n(x \in W_{e,n}))}$, otherwise it says "no" (f is the function from Lemma 11(b) which witnesses that A is promptly simple). Of course this oracle gives the correct answer for all $x \in W_e \cap \bar{A}$. There are in general infinitely many $x \in W_e \cap A$ where the oracle gives the wrong answer. But if W_e is infinite then there are as well infinitely many $x \in W_e \cap A$ where the oracle gives the correct answer. This enables us to prove the following Lemma 16.

We use the same notation as in Soare [4], [6]. A full e-state ν is a triple $\langle e, \sigma, \tau \rangle$ with $\sigma, \tau \subseteq \{0, \ldots, e\}$. If $\nu = \langle e, \sigma, \tau \rangle$ and $\nu' = \langle e, \sigma', \tau' \rangle$ are full e-states then ν covers ν' (written $\nu \geq \nu'$) if $\sigma \supseteq \sigma'$ and $\tau \subseteq \tau'$. $A =^* B$ means that the sets A and B have a finite difference. $\mathscr{L}(A)$ is the lattice of r.e. sets W with $A \subseteq W$.

LEMMA 16. *Let $(U_e)_{e \in \omega}$, $(\hat{U}_e)_{e \in \omega}$, $(V_e)_{e \in \omega}$, $(\hat{V}_e)_{e \in \omega}$ be recursive arrays of r.e. sets with recursive enumerations $(U_{e,n})_{e,n \in \omega} \ldots$ (i.e. the set $\{\langle e, x \rangle \mid x \in U_e\}$ is r.e. and the function $n \mapsto \{\langle e, x \rangle \mid x \in U_{e,n}\}$ has only finite sets as values and is recursive and nondecreasing; analogous for the other sets).*

Further let A, $\hat{A}$ be promptly simple sets with recursive enumerations $(A_n)_{n \in \omega}$, $(\hat{A}_n)_{n \in \omega}$.

Define for each full e-state $\nu = \langle e, \sigma, \tau \rangle$ the sets

$$D_\nu^A := \{x \mid \exists n(x \in A_{n+1} - A_n \wedge \forall i \leq e(x \in U_{i,n} \leftrightarrow i \in \sigma) \wedge \forall i \leq e\, (x \in \hat{V}_{i,n} \leftrightarrow i \in \tau))\}$$

and

$$D_\nu^{\hat{A}} := \{x \mid \exists n(x \in \hat{A}_{n+1} - \hat{A}_n \wedge \forall i \leq e\, (x \in \hat{U}_{i,n} \leftrightarrow i \in \sigma) \wedge \forall i \leq e\, (x \in V_{i,n} \leftrightarrow i \in \tau))\}.$$

Assume that the following "weak covering property" holds: if ν is a full e-state such that $D_\nu^{\hat{A}}$ is infinite then there is some full e-state $\nu' = \langle e, \sigma', \tau' \rangle \geq \nu$ with

$$C_{\nu'}^A := \{x \mid \exists n(x \notin A_n \wedge \forall i \leq e\, (x \in U_{i,n} \leftrightarrow i \in \sigma') \wedge \forall i \leq e\, (x \in \hat{V}_{i,n} \leftrightarrow i \in \tau'))\}$$

infinite, and if D_ν^A is infinite then there is some full e-state $\nu' = \langle e, \sigma', \tau' \rangle \leqslant \nu$ with

$$C_{\nu'}^{\hat{A}} := \{x \mid \exists n(x \notin \hat{A}_n \wedge \forall i \leq e(x \in \hat{U}_{i,n} \leftrightarrow i \in \sigma') \wedge \forall i \leq e(x \in V_{i,n} \leftrightarrow i \in \tau'))\}$$

infinite. Then there are recursive arrays $(\hat{U}_e^+)_{e \in \omega}$, $(\hat{V}_e^+)_{e \in \omega}$ with recursive enumerations $(\hat{U}_{e,n}^+)_{e,n \in \omega}$, $(\hat{V}_{e,n}^+)_{e,n \in \omega}$ such that for every e, n,

$$\hat{U}_{e,n}^+ \subseteq \hat{U}_{e,n}, \quad \hat{V}_{e,n}^+ \subseteq \hat{V}_{e,n},$$
$$\hat{U}_e^+ \cap \bar{\hat{A}} = \hat{U}_e \cap \bar{\hat{A}}, \quad \hat{V}_e^+ \cap \bar{A} = \hat{V}_e \cap \bar{A}$$

and such that for the arrays $(U_e)_{e\in\omega}$, $(\hat{U}_e^+)_{e\in\omega}$, $(V_e)_{e\in\omega}$, $(\hat{V}_e^+)_{e\in\omega}$ *the following "strong covering property" holds (where* $\tilde{D}_\nu^A$, $\tilde{D}_\nu^{\hat{A}}$ *are defined as* D_ν^A, $D_\nu^{\hat{A}}$ *with* $\hat{V}_{i,n}^+$ *instead of* $\hat{V}_{i,n}$, *respectively,* $\hat{U}_{i,n}^+$ *instead of* $\hat{U}_{i,n}$*):*

$$\forall \nu(\tilde{D}_\nu^{\hat{A}} \text{ infinite} \Rightarrow \exists \nu' \geq \nu\ (\tilde{D}_{\nu'}^{A} \text{ infinite}))$$

and

$$\forall \nu(\tilde{D}_\nu^{A} \text{ infinite} \Rightarrow \exists \nu' \leq \nu\ (\tilde{D}_{\nu'}^{\hat{A}} \text{ infinite})).$$

Further we get for every e, x, n,

$$x \in A_n \cap \hat{V}_e^+ \to n > 0 \wedge x \in \hat{V}_{e,n-1}^+$$

and

$$x \in \bar{A}_n \cap \hat{U}_e^+ \to n > 0 \wedge x \in \hat{U}_{e,n-1}^+.$$

Proof. Choose an index i for the promptly simple set A and a recursive function f as in Lemma 11(b). Define a recursive function by $h(n) := \mu m(W_{i,n} \subseteq A_m)$. Then for every infinite set W_e there are infinitely many $x \in W_e$ with $x \in A_{h(f(\mu n(x\in W_{e,n})))}$. Fix a total recursive function q such that for every full e-state ν we have $W_{q(\nu)} = C_\nu^A$ (C_ν^A as in the claim of this lemma).

For every $j \in \omega$ we enumerate the set $\hat{V}_j^+$ as follows:

Step n. If $x \in \hat{V}_{j,n}$ and $x \notin \hat{V}_{j,n'}$ for all $n' < n$ and $x \notin A_n$ and $x \notin A_{h(f(\mu k(x\in W_{q(\nu),k})))}$ for every full e-state $\nu = \langle e, \sigma, \tau\rangle$, $e \leq x$, such that for some $m \leq n$

$$\forall i \leq e\ (x \in U_{i,m} \leftrightarrow i \in \sigma) \wedge \forall i \leq e(x \in \hat{V}_{i,m} \leftrightarrow i \in \tau),$$

then enumerate x in $\hat{V}_j^+$.

Let $\hat{V}_{j,n}^+$ be the set of elements which are enumerated in $\hat{V}_j^+$ by Step n.

In order to define $\hat{U}_j^+$ we fix a recursive function $\hat{q}$ such that for every ν, $W_{\hat{q}(\nu)} = C_\nu^{\hat{A}}$. The functions $\hat{h}$, $\hat{f}$ are chosen for $\hat{A}$ analogously as h, f for A.

Step n. If $x \in \hat{U}_{j,n}$ and $x \notin \hat{U}_{j,n'}$ for all $n' < n$ and $x \notin \hat{A}_n$ and $x \notin \hat{A}_{\hat{h}(\hat{f}\mu k(x\in W_{\hat{q}(\nu),k}))}$ for every full e-state $\nu = \langle e, \sigma, \tau\rangle$, $e \leq x$, such that for some $m \leq n$,

$$\forall i \leq e(x \in \hat{U}_{i,m} \leftrightarrow i \in \sigma) \wedge \forall i \leq e(x \in V_{i,m} \leftrightarrow i \in \tau),$$

then enumerate x in $\hat{U}_j^+$.

Let $\hat{U}_{j,n}^+$ be the set of elements which are enumerated in $\hat{U}_j^+$ by Step n.

With these definitions all claims except the "strong covering property" are obvious.

Assume $\tilde{D}_\nu^{\hat{A}}$ is infinite for some full e-state ν. Since $\hat{U}_{j,n}^+ \subseteq \hat{U}_{j,n}$ for every j, n this implies that $D_{\tilde{\nu}}^{\hat{A}}$ is infinite for some $\tilde{\nu} \geq \nu$. By our assumption there is some $\nu' \geq \tilde{\nu}$ such that $C_{\nu'}^A$ is infinite. By our choice for h, f the set

$$S := \{x \geq e \mid x \in C_{\nu'}^A \wedge x \in A_{h(f(\mu k(x\in W_{q(\nu'),k})))}\}$$

is then as well infinite. We show that $S \subseteq \bigcup\{\tilde{D}_{\nu''}^A \mid \nu'' \geq \nu'\}$. Since there are only finitely many $\nu'' \geq \nu'$ this is obviously enough. Thus assume that $x \in S$ and fix some n_0 such that

$$x \notin A_{n_0} \wedge \forall i \leq e(x \in U_{i,n_0} \leftrightarrow i \in \sigma')$$
$$\wedge \forall i \leq e(x \in \hat{V}_{i,n_0} \leftrightarrow i \in \tau')$$

where $\nu' = \langle e, \sigma', \tau' \rangle$. Then there is some $\tau'' \subseteq \tau'$ such that $\forall i \leq e\, (x \in \hat{V}^+_{i,n_0} \leftrightarrow i \in \tau'')$ since for every i, $\hat{V}^+_{i,n_0} \subseteq \hat{V}_{i,n_0}$. From the definition it is clear that $x \notin \hat{V}^+_{j,n+1} - \hat{V}^+_{j,n}$ for every j and every $n \geq n_0$ (consider $m := n_0$ and $\nu := \nu'$ in the definition of $\hat{V}^+_{j,n+1}$). Therefore $x \in \tilde{D}^A_{\nu''}$ with $\nu'' = \langle e, \sigma'', \tau'' \rangle$ for some $\sigma'' \supseteq \sigma'$.

The second part of the "strong covering property" is symmetrical.

THEOREM 17. *For any two promptly simple sets A and $\hat{A}$ with semilow complements there is an automorphism Φ of $\mathscr{E}$ such that $\Phi(A) = \hat{A}$.*

PROOF. We construct the automorphism in three steps.

First we use the fact that $\bar{A}$ and $\bar{\hat{A}}$ are semilow. A slight variation of the construction in Soare [6] gives us recursive arrays $(U_n)_{n\in\omega}$, $(\hat{U}_n)_{n\in\omega}$, $(V_n)_{n\in\omega}$, $(\hat{V}_n)_{n\in\omega}$ together with enumerations so that the "weak covering property" in Lemma 16 is satisfied. We have, further, for all n, $W_n =^* U_n$ and $W_n =^* V_n$. The sets $\hat{U}_n$, $\hat{V}_n$ have the property that $\hat{U}_n \cap \bar{\hat{A}}$ can be used as $\Phi(U_n) \cap \bar{\hat{A}}$ and $\hat{V}_n \cap \bar{A}$ can be used as $\Phi(V_n) \cap \bar{A}$ (except for changes at finitely many points) for the constructed automorphism Φ.

Then we use the assumption that A and $\hat{A}$ are promptly simple in order to shrink the parts $\hat{U}_n \cap \hat{A}$, $\hat{V}_n \cap A$ of the sets $\hat{U}_n$, respectively $\hat{V}_n$, in Lemma 16. The corrected images $U^+_n \subseteq \hat{U}_n$ and $V^+_n \subseteq \hat{V}_n$ have the property that their intersection with $\hat{A}$, respectively A, is small enough so that there is no a priori obstacle to a possible extension of these parts of the sets to images $\Phi(U_n) \cap \hat{A}$, respectively $\Phi^{-1}(V_n) \cap A$, for the constructed automorphism Φ. The lacking of an a priori obstacle is expressed by the "strong covering property" (see conclusion of Lemma 16).

Finally we apply the Extension Theorem of Soare [4]. This theorem says that if the "strong covering property" holds then we can in fact extend the sets $U^+_n \cap \hat{A}$ and $V^+_n \cap A$ to suitable images $\Phi(U_n) \cap \hat{A}$, respectively $\Phi^{-1}(V_n) \cap A$.

In the following we describe some details of the sketched procedure. Soare constructs in [6] for a semilow set $\bar{A}$ an isomorphism between $\mathscr{L}(A)$ and $\mathscr{E}$ (the framework of the construction is very similar to the one in [4] for the proof of the Extension Theorem).

In Soare's construction [6] one plays simultaneously on two pinball machines M and $\hat{M}$. If an element is enumerated into A one removes it forever from M but one never removes an element from $\hat{M}$. After the play one can define a function which maps $\bar{A}$ (the set of elements which remain in M) one-one onto ω (the set of elements which remain in $\hat{M}$) in such a way that this function induces an isomorphism between $\mathscr{L}(A)$ and $\mathscr{E}$.

We change $\hat{M}$ in such a way that it plays with respect to $\hat{A}$ the same role as M does with respect to A. Thus only the elements of $\bar{\hat{A}}$ remain in $\hat{M}$ in the end and we get a function q which maps $\bar{A}$ one-one onto $\bar{\hat{A}}$ and induces an isomorphism between $\mathscr{L}(A)$ and $\mathscr{L}(\hat{A})$.

More exactly one constructs in this changed version recursive arrays of r.e. sets $(U_e)_{e\in\omega}$, $(\hat{U}_e)_{e\in\omega}$, $(V_e)_{e\in\omega}$, $(\hat{V}_e)_{e\in\omega}$ together with recursive enumerations $(U_{e,n})_{e,n\in\omega}$, $(\hat{U}_{e,n})_{e,n\in\omega}$, $(V_{e,n})_{e,n\in\omega}$, $(\hat{V}_{e,n})_{e,n\in\omega}$ of these sets (in the sense of Lemma 16) and recursive enumerations $(A_n)_{n\in\omega}$, $(\hat{A}_n)_{n\in\omega}$ of the sets A, $\hat{A}$ such that for every $e > 1$, $W_e =^* U_e$ and $W_e =^* V_e$, such that there is a one-one function q from $\bar{A}$ onto $\bar{\hat{A}}$ with $q[U_e \cap \bar{A}] =^* \hat{U}_e \cap \bar{\hat{A}}$ and $q^{-1}[V_e \cap \bar{\hat{A}}] =^* \hat{V}_e \cap \bar{A}$ for every $e \in \omega$ (use the method of Theorem 1.3 in [4] in order to define q) and such that in addition the "weak covering property" from Lemma 16 is satisfied. Observe that another (trivial) change in Soare's construction [6] is needed in order to make sure that for every $e > 1$, $W_e =^* U_e$ and $W_e =^* V_e$. In Case 4(c) of the construction one enumerates x in U_e (V_e) even if $x \in U_{1,s}$ ($V_{1,s}$).

The enumerations are determined as follows: For any set X one defines X_n as the set of elements which are enumerated in X by the end of stage n of the pinball machine construction. Thus the finite sets $U_{e,n}$, $\hat{U}_{e,n}$, $V_{e,n}$, $\hat{V}_{e,n}$ are determined for every e, $n \in \omega$. As in [6] we have $U_1 = A$ and in our changed version we have in addition $V_1 = \hat{A}$. Define $A_n := U_{1,n}$ and $\hat{A}_n := V_{1,n}$ for every n.

An analog to the "weak covering property" is not explicitly stated in [6]. But one can easily put it together from the covering properties which are proved in [6] and which hold as well for our changed version (with the same proofs). An element $\hat{x}$ can be enumerated into $\hat{A}$ only according to Case 4(b) or Case 2(a) of the construction. In the first case the element $\hat{x}$ comes directly from pocket $\hat{P}$ or $\hat{Q}$. In the latter case $\hat{x}$ is sitting above hole $\hat{H}_1$ or $\hat{H}_2$ before it is enumerated into $\hat{A}$. We need not analyze the trivial case of hole $\hat{H}_1$. If $\hat{x}$ sits above hole $\hat{H}_2$ at the end of stage s then there is some stage $t \le s$ such that $\hat{x}$ was in $\hat{P}$ or $\hat{Q}$ at stage t in some state $\nu' \ge \nu(s, x, \hat{x})$. Therefore if for the previous definition $D_\nu^{\hat{A}}$ is infinite for some ν there is some $\tilde{\nu} \ge \nu$ such that $\tilde{\nu}$ occurs infinitely often in stream $\hat{D}$ or in stream $\hat{Q}$. By Lemma 6.6 of Soare [6] the stream C covers every stream $\hat{X}$ of $\hat{M}$. Thus there is some $\nu' \ge \tilde{\nu}$ which occurs infinitely often in stream C. For this state $\nu' \ge \nu$ the set $C_{\nu'}^{\hat{A}}$ is infinite.

We have now shown that all assumptions of Lemma 16 are satisfied for the previously specified sets and enumerations. Let $(\hat{U}_e^+)_{e\in\omega}$, $(\hat{V}_e^+)_{e\in\omega}$ be the recursive arrays with enumerations $(\hat{U}_{e,n}^+)_{e,n\in\omega}$, $(\hat{V}_{e,n}^+)_{e,n\in\omega}$ which are given by the conclusion of Lemma 16. We show that the constructed sets satisfy all assumptions of the Extension Theorem (Theorem 2.2 in [4], with $\hat{A}$ instead of B).

We have to specify a simultaneous enumeration of the sets A, $\hat{A}$, $(U_e)_{e\in\omega}$, $(V_e)_{e\in\omega}$, $(\hat{U}_e^+)_{e\in\omega}$, $(\hat{V}_e^+)_{e\in\omega}$ which puts at every step exactly one element in exactly one of these sets, without repetition (this is the notion of simultaneous enumeration which is used in [4, p. 82]). In order to get this we consider the given enumeration of these sets. For every n we first enumerate all elements of $A_{n+1} - A_n$ in A, in increasing order. Then all elements of $\hat{A}_{n+1} - \hat{A}_n$ are enumerated in $\hat{A}$ in increasing order. Then we proceed analogously for all the sets $U_0, U_1, \ldots, V_0, V_1 \ldots, \hat{U}_0^+, \hat{U}_1^+, \ldots, \hat{V}_0^+, \hat{V}_1^+, \ldots$ (in this order). After we have finished this we go to $n + 1$ and enumerate all elements of $A_{n+2} - A_{n+1}$, etc.

For this new enumeration we have for every e, $A \searrow \hat{V}_e^+ = \emptyset$ and $\hat{A} \searrow \hat{U}_e^+ = \emptyset$ (by the analogous properties of the old enumeration). Here $U \searrow V$ is defined as the set of elements which are first enumerated into U and afterwards into V.

Further if one defines for the new enumeration the sets D_ν^A, $D_\nu^{\hat{A}}$ as in the assumption of the Extension Theorem, then we have $D_\nu^A = \tilde{D}_\nu^A$ and $D_\nu^{\hat{A}} = \tilde{D}_\nu^{\hat{A}}$, where $\tilde{D}_\nu^A$, $\tilde{D}_\nu^{\hat{A}}$ are the sets which are defined in Lemma 16 (for the old enumeration). Thus Lemma 16 asserts that the covering property in the assumption of the Extension Theorem is satisfied.

By the Extension Theorem there exist r.e. sets $\hat{U}_e^- \subseteq \hat{A}$, $\hat{V}_e^- \subseteq A$ and a function p which maps A one-one onto $\hat{A}$ such that for every e, $p[A \cap U_e] =^* \hat{A} \cap (\hat{U}_e^+ \cup \hat{U}_e^-)$ and $p^{-1}[\hat{A} \cap V_e] =^* A \cap (\hat{V}_e^+ \cup \hat{V}_e^-)$.

We combine p with the function q from the first part of the proof in order to define a permutation r of ω: Set

$$r(x) := \begin{cases} p(x) & \text{if } x \in A, \\ q(x) & \text{if } x \in \bar{A}. \end{cases}$$

Define the desired map Φ by $\Phi(W) := r[W]$ for r.e. sets W. Obviously we have $\Phi(A) = r[A] = \hat{A}$. Further for any r.e. W take some $e > 1$ with $W_e = W$. Then

$$\Phi(W) = r[W] = r[W_e] =^* r[U_e] =^* \hat{U}_e^+ \cup \hat{U}_e^-.$$

Therefore the set $\Phi(W)$ is in fact r.e. Finally for a given r.e. set V we take some $e > 1$ with $W_e = V$. Since then $r[\hat{V}_e^+ \cup \hat{V}_e^-] =^* V_e =^* W_e$ we see that there is an r.e. set W with $\Phi(W) = V$. It is obvious that for all r.e. sets V, W, $V \subseteq W \Leftrightarrow \Phi(V) \subseteq \Phi(W)$. Thus Φ is an automorphism of $\mathscr{E}$.

COROLLARY 18. *All r.e. generic sets are automorphic. Thus they all have the same properties in the lattice of r.e. sets.*

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