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RECURSIVELY INVARIANT β-RECURSION THEORY

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We introduce recursively invariant β-recursion theory as a new approach towards recursion theory on an arbitrary limit ordinal β. We follow Friedman and Sacks and call a subset of β β-recursively enumerable if it is Σ_1-definable over L_β. Since Friedman-Sacks' notion of a β-finite set is not invariant under β-recursive permutations of β we turn to a different notion. Under all possible invariant generalizations of finite there is a canonical one which we call i-finite. We consider further in the inadmissible case those criteria for the adequacy of generalizations of finite which have earlier been developed by Kreisel, Moschovakis and others. We look at infinitary languages over inadmissible sets L_β and the compactness theorem for these languages, the characterization of the basic notions of β-recursion theory in terms of model theoretic invariance, the definition of β-recursion theory via an equation calculus and axioms for computation theories. In turns out that in all these approaches the i-finite sets are those subsets of β respectively L_β which behave like finite sets.

Invariant β-recursion theory contains classical recursion theory and α-recursion theory as special cases. We start in the second half of this paper the systematic development of invariant β-recursion theory for all limit ordinals β. We study in particular i-degrees, which generalize Turing degrees and α-degrees. Besides 0 (the degree of β-recursive sets) and $0'$ (the largest β-r.e. degree) there exist incomparable β-r.e. i-degrees for every limit ordinal β. Similar as the step from ω to α gave rise to the introduction of regular respectively hyperregular sets we arrive in invariant β-recursion theory at the new notion of an i-absolute β-r.e. set. This notion is useful in order to describe a difference among hyperregular β-r.e. sets which occurs exclusively in the inadmissible case. The study of i-degrees is most difficult for those β which are strongly inadmissible (i.e. $\sigma 1 \operatorname{cf} \beta < \beta^*$). For those strongly inadmissible β where β^* is regular we give two new constructions which rely heavily on the combinatorial properties of regular cardinals ($\diamond$, closed unbounded sets and the Δ-System lemma). We construct a β-r.e. degree $\boldsymbol{a} > \boldsymbol{0}$ such that no degree $\boldsymbol{b} \leq \boldsymbol{a}$ contains a simple set and we prove a splitting theorem for simple β-r.e. sets. We base the definition of a simple set on the general notion of a l-finite set.

1. Introduction and foundations

β is always a limit ordinal in the following. We want to study recursion theory on β. It is convenient to have a domain which does not contain ordinals only. The elements of β are not even closed under pairing $x, y \to \{x, y\}$. Therefore we take as domain instead of β a slightly larger collection of sets which we can build up mechanically on our way through β. A very natural choice for such a domain is

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L_β — the collection of all sets which are constructible on levels less than β. L_β is generated by iterating β times the operation

$$u \to \mathrm{Def}(u)$$

where

$$\mathrm{Def}(u) := \{x \subseteq u \mid x \text{ is definable over } \langle u, \in \restriction u \times u\rangle \text{ by some first order formula with parameters from } u\}.$$

Define

$$L_0 = \emptyset,$$
$$L_{\gamma+1} = \mathrm{Def}(L_\gamma),$$
$$L_\lambda = \bigcup \{L_\gamma \mid \gamma < \lambda\} \quad \text{for limit ordinals } \lambda.$$

Every L_γ is transitive, the ordinals in L_γ are exactly the ordinals less than γ and for $\gamma < \delta$ we have $L_\gamma \subsetneq L_\delta$. Further the function $\gamma \mapsto L_\gamma$ from β into L_β is $\Sigma_1 L_\beta$ definable for every limit ordinal β (see Devlin [3] or [4] for details concerning the constructible hierarchy).

Definition 1.1. A set $A \subseteq L_\beta$ is *β-recursively enumerable* (β-r.e.) iff A is $\Sigma_1 L_\beta$ (i.e. A is definable over L_β by some Σ_1-formula which may contain elements of L_β as parameters).

The definition of a β-r.e. set and a β-recursive function is due to Friedman and Sacks [8].

A canonical enumeration procedure is associated with every Σ_1-definition $\varphi(x)$ of a β-r.e. set A:

Generate successively the levels $L_0, L_1, \ldots, L_\gamma, \ldots (\gamma < \beta)$ of the constructible hierarchy up to β. Enumerate at every step γ those elements z into A which satisfy $L_\gamma \models \varphi(z)$ and which have not already been enumerated before.

This gives an enumeration of the set A because we have for every $z \in L_\beta$

$$L_\beta \models \varphi(z) \Leftrightarrow \exists \gamma < \beta \, (L_\gamma \models \varphi(z)),$$

using the fact that φ is a Σ_1-formula.

A β-r.e. set together with the described enumeration procedure is a perfect example for the general concept of a recursively enumerable set. This concept was explicated e.g. by Post [34] and Sacks [39]. They describe a recursively enumerable set as a generated set. The set is generated by a predetermined effective process which puts at certain steps elements into the set. Once an element is placed in the set, it stays there. We follow Sacks [39] and speak of a RE set if we want to appeal to this general concept. The example shows that the concept of a RE set does not require any strong closure conditions of the considered domain (like e.g. admissibility).

Definition 1.2. A partial function $f: L_\beta \to L_\beta$ is called *partial β-recursive* iff the graph of f is β-r.e. If f is in addition defined on all elements of L_β, then we call f β-recursive.

A canonical way to compute a partial β-recursive function f comes together with its definition. We fix a $\Sigma_1 L_\beta$ definition of the graph of f. Given $x \in L_\beta$ we start to enumerate the graph of f step by step as described before. At every step γ we check whether some pair of the form $\langle x, z\rangle$ occurs among the enumerated elements. If we find such a pair $\langle x, z\rangle$ we say that the computation of $f(x)$ converges (at step γ) with value z.

At this point the question arises which properties of subsets of L_β one should study in β-recursion theory. The second basic concept of recursion theory is finiteness. Therefore above all we have to find out which sets are playing the role of finite sets in β-recursion theory. We choose here the principle of recursive invariance as our guide.

In mathematics the study of invariant properties was first formulated as a general program by Felix Klein (Erlanger Programm, 1872). Felix Klein suggested to define branches of mathematics in terms of a space X (i.e. a set X) and a group G of transformations acting on that space (i.e. a set of permutations of X which is a group under the law of composition $(f, g) \mapsto f \circ g$). A property of subsets of X is called G-invariant if for every set $A \subseteq X$ and every $f \in G$ we have that A has this property iff $f[A]$ has this property. The branch of mathematics determined by X and G is the study of G-invariant properties.

Klein's program has penetrated large parts of modern mathematics. It was introduced into (ordinary) recursion theory by Rogers [35]. Here X is the set of natural numbers and G is the group of all recursive permutations of the natural numbers. Instead of G-invariant one says recursively invariant. All important notions of ordinary recursion theory (except subrecursive hierarchies etc.) are recursively invariant. In fact Rogers [35] states: 'The notion of recursive invariance characterizes our theory and serves as a touchstone for determining possible usefulness of new concepts.'

It seems that Klein's program was never explicitly mentioned in generalized recursion theory. Nevertheless one followed it intuitively. For example in α-recursion theory and in recursion theory in higher types all the considered notions are invariant under the approximate group of recursive permutations.

In β-recursion theory the definitions of a β-recursively enumerable set and of a β-recursive function are very convincing. In fact these are the only definitions in β-recursion theory, which are immediately justified by our intuition. Since there is a canonical choice of X and G we can use Klein's program as a guide for the definition of further notions in β-recursion theory.

Definition 1.3. A property of subsets of L_β is called *recursively invariant* iff it is G-invariant, where G is the group of all β-recursive permutations of L_β.

We want to study, which subsets of L_β are playing the role of finite sets. A 'finite' set in β-recursion theory should be β-recursive and bounded (call a set $M \subseteq L_\beta$ bounded iff $M \subseteq L_\gamma$ for some $\gamma < \beta$). Thus we consider

$$C := \{Q \subseteq \mathfrak{P}(L_\beta) \mid Q \text{ is recursively invariant and every element of } Q \text{ is } \beta\text{-recursive and bounded}\}.$$

The union of any number of elements of C is again an element of C. Therefore C has a largest element which we call I. This largest element is the most interesting one from the mathematical point of view. More important: I is distinguished from all the other elements of C through its coherence with the notion of a β-recursively enumerable set. We can make this point more precise after Theorem 1.11, where we have a perspicuous characterization of I at hand. It is convenient to prove some other fundamental facts first.

Definition 1.4. A subset of L_β is called *i-finite* ('invariantly finite') iff it is an element of I—the largest recursively invariant class of β-recursive bounded subsets of L_β.

Sometimes we write i_β-finite instead of i-finite in order to stress the dependence upon β. Observe that for $\beta = \omega$ the i-finite sets are exactly the finite sets and for $\beta = \alpha$ (α admissible) the i-finite sets are exactly the α-finite sets. Thus ordinary recursion theory and α-recursion theory are special cases of recursively invariant β-recursion theory.

Small greek letters will always denote ordinals in the following.

Definition 1.5. (a) $\beta^* := \mu\delta \leq \beta$ (there is a β-recursive function which maps β one-one into δ). β^* is called the *Σ_1-projection of β.*

(b) $\sigma 1 \operatorname{cf} \beta := \mu\delta \leq \beta$ (there is a β-recursive function which maps δ cofinally into β). $\sigma 1 \operatorname{cf} \beta$ is called the *recursive cofinality of β.*

(c) An ordinal $\gamma < \beta$ is called a *β-cardinal* iff

$$L_\beta \models [\neg \exists\, \delta < \gamma\ \exists f\ (f \text{ maps } \gamma \text{ one-one into } \delta)].$$

(d) $\beta\text{-card}(x) := \mu\delta$ (there is some $f \in L_\beta$ which maps x one-one into δ) for any $x \in L_\beta$. $\beta\text{-card}(x)$ is called the *β-cardinality of x.*

Observe that β^* and $\sigma 1 \operatorname{cf} \beta$ are always β-cardinals. β is admissible iff $\sigma 1 \operatorname{cf} \beta = \beta$. A Skolem hull argument as in the proof of the following lemma shows that there is always a largest β-cardinal if β is inadmissible [9]. This implies that $\beta^* < \beta$ for inadmissible β. It is easy to see that $\beta\text{-card}(x)$ is a well-defined ordinal less than β for every $x \in L_\beta$.

The following lemma is well known. Its proof is a refinement of a standard proof of GCH in L (see [3, 4]). '$L \models \text{GCH}$' follows from the lemma as a special case (take $\beta = \infty$).

Lemma 1.6 (reflection principle). *Assume that ρ is a β-cardinal and $\rho \leq \beta^*$. Further assume that $x \in L_\beta$ and $x \subseteq L_\delta$ for some $\delta < \rho$. Then $x \in L_\rho$.*

Proof. Assume $\delta \geq \omega$ (otherwise trivial). Let h be a Σ_1 Skolem function for L_β without parameters (see [4]). Define the set D as the closure of $L_{\delta+1} \cup \{x\}$ under the pairing function $u, v \to \{u, v\}$. Define $Y := h[\omega \times D]$. Then $D \subseteq Y \prec_{\Sigma_1} L_\beta$ and the transitive collapse of Y is some L_γ with a limit ordinal $\gamma \leq \beta$ according to Devlin [4]. Call the collapsing function π. Since $x = \pi(x) \in L_\gamma$ it is enough to show that $\gamma \leq \rho$. $h \restriction \omega \times D$ is Σ_1 definable over L_β with parameters from $L_{\delta+1} \cup \{x\}$. $(h \restriction \omega \times D)^{-1}$ is in general not a function. Therefore we apply the canonical Σ_1 uniformization procedure (see [4]) and get a one-one function $f \subseteq (h \restriction \omega \times D)^{-1}$ with dom $f = Y$. Furthermore f is definable over L_β (and over Y) by some Σ_1 formula ψ with parameters from $L_{\delta+1} \cup \{x\}$. D is transitive and therefore the same formula ψ defines over L_γ a function $\tilde{f}$ which maps L_γ one-one into D where $\tilde{f}(\pi(u)) = \pi(f(u))$. This implies $\gamma < \beta$ since one can map D β-recursively one-one into δ and according to our assumption we have $\delta < \beta^*$. Therefore we know that $\tilde{f} \in L_\beta$ and there is an other element of L_β which maps D one-one into δ. Since ρ is a β-cardinal this implies $\gamma < \rho$. □

Corollary 1.7. *Assume that $x \in L_\beta$ and $x \subseteq \beta^*$. Then there is a $\delta \leq \beta^*$ and a function $f \in L_\beta$ which maps δ one-one onto x.*

Proof. f is the function which enumerates x in order. For admissible β it is obvious that $f \in L_\beta$. Otherwise we know that $\beta^* < \beta$ and we show inductively that $f \restriction \sigma \in L_{\beta^*}$ for every $\sigma < \delta$. For limit ordinals σ we use in this induction the preceding Lemma 1.6. □

Remark 1.8. Without the assumption $\rho \leq \beta^*$ in Lemma 1.6 one comes into difficulties in the proof of Lemma 1.6 in the case that β is not a limit of limit ordinals. If one works with Jensen's J-hierarchy one can use at this point Jensen's uniformization theorem (see [17]). We can avoid here all complicated machinery because only the following two facts are needed (they are derived in the proof of Theorem 1.11):

(a) for every i-finite set $x \in L_\beta$ there is some $f \in L_\beta$ which maps x one-one onto some $\gamma < \sigma 1 \operatorname{cf} \beta$

(b) if $x \in L_\beta$ is not i-finite, then there is some $f \in L_\beta$ which maps $\sigma 1 \operatorname{cf} \beta$ one-one into x.

Lemma 1.9. *Assume $A \subseteq L_\beta$ is β-r.e. Then there is an ordinal $\delta \leq \beta$ and a β-recursive function f which maps δ one-one onto A. We can always choose $\delta \leq \max(\beta^*, \sigma 1 \operatorname{cf} \beta)$.*

Proof. According to Devlin [4] there exists a β-recursive function which maps β onto L_β. One can apply Σ_1-uniformization to the inverse of this function and thus get a β-recursive function which maps L_β one-one into β. Therefore we can assume that the given β-r.e. set A is a subset of β^*.

Fix a $\Sigma_1 L_\beta$ definition φ of A and a β-recursive cofinal continuous increasing function $q:\sigma 1 \operatorname{cf} \beta \to \beta$. One can define a β-recursive function $h:\sigma 1 \operatorname{cf}\beta \to L_\beta$ such that $h(\gamma)$ is a function which maps some $\delta_\gamma \leq \beta^*$ one-one onto

$$\{x \mid L_{q(\gamma+1)} \models [\varphi(x)] \wedge \neg L_{q(\gamma)} \models [\varphi(x)]\}$$

for every $\gamma < \sigma 1 \operatorname{cf}\beta$ ($h(\gamma)$ exists by Corollary 1.7). It is easy to define a β-recursive function g which maps some $\delta \leq \max(\beta^*, \sigma 1 \operatorname{cf}\beta)$ one-one onto $\bigcup \{\{\gamma\} \times \delta_\gamma \mid \gamma < \sigma 1 \operatorname{cf}\beta\}$ (use Lemma 1.6 if $\sigma 1 \operatorname{cf}\beta < \beta^*$). We get the wanted function f by combining g and h. □

Corollary 1.10. *There exists a β-recursive function which maps $\max(\beta^*, \sigma 1 \operatorname{cf}\beta)$ one-one onto β(see [9]) and there exists a β-recursive function which maps β one-one onto L_β.*

Proof. By Lemma 1.9 there exists a β-recursive function which maps some $\delta \leq \max(\beta^*, \sigma 1 \operatorname{cf}\beta)$ one-one onto β (take $A := \beta$ in the lemma). By definition of β^* and $\sigma 1 \operatorname{cf}\beta$ we have $\delta \geq \beta^*$ and $\delta \geq \sigma 1 \operatorname{cf}\beta$. Therefore $\delta = \max(\beta^*, \sigma 1 \operatorname{cf}\beta)$.

For the second part of the corollary we take $A := L_\beta$ in Lemma 1.9. As before we get a β-recursive function which maps $\max(\beta^*, \sigma 1 \operatorname{cf}\beta)$ one-one onto L_β. We combine this function with the preceding one in order to get a β-recursive function which maps β one-one onto L_β. □

Theorem 1.11. *The set I of i-finite subsets of L_β is a β-recursive subset of L_β. One has for every $x \in L_\beta$:*

$$\begin{aligned} x \text{ is i-finite} &\Leftrightarrow \beta\text{-card}(x) < \sigma 1 \operatorname{cf}\beta \\ &\Leftrightarrow \exists f \in L_\beta\ \exists \delta < \sigma 1 \operatorname{cf}\beta\ (f \text{ maps } x \text{ one-one onto } \delta). \end{aligned}$$

Proof. Take an i-finite set x. Since x is β-r.e. there exists by Lemma 1.9 an ordinal $\delta \leq \beta$ and a β-recursive function f which maps δ one-one onto x.

Assume for a contradiction that $\delta \geq \sigma 1 \operatorname{cf}\beta$. Define then $\tilde{x} := f[\sigma 1 \operatorname{cf}\beta]$. Fix a β-recursive cofinal increasing function $q:\sigma 1 \operatorname{cf}\beta \to \beta$ such that $\tilde{x} \cap q[\sigma 1 \operatorname{cf}\beta] = \emptyset$. Define a β-recursive permutation h of L_β as follows:

$$h(z) := \begin{cases} q(f^{-1}(z)) & \text{if } z \in \tilde{x}, \\ f(q^{-1}(z)) & \text{if } z \in q[\sigma 1 \operatorname{cf}\beta], \\ z & \text{otherwise.} \end{cases}$$

$\tilde{x}$ is β-recursive (since x is β-recursive) and $q[\sigma 1 \operatorname{cf}\beta]$ is β-recursive (since q is increasing and cofinal). Thus h is β-recursive. $h(x)$ is unbounded and therefore not an element of I. This is a contradiction to the recursive invariance of I.

Since $\delta < \sigma 1 \operatorname{cf}\beta$ it is clear that $f \in L_\beta$.

Further one verifies easily that $\{x \in L_\beta \mid \beta\text{-card}(x) < \sigma 1 \operatorname{cf} \beta\}$ is recursively invariant. Thus it is a subset of I by the definition of I. This finishes the proof of the claimed equalities.

The set $\{x \in L_\beta \mid \beta\text{-card}(x) < \sigma 1 \operatorname{cf} \beta\}$ is obviously β-r.e. The set $S := \{x \in L_\beta \mid \exists f \in L_\beta$ (f maps $\sigma 1 \operatorname{cf} \beta$ one-one into x)$\}$ is as well β-r.e. Therefore we know that I is β-recursive as soon as we have shown that $S = L_\beta - I$.

We have $S \subseteq L_\beta - I$ since $\sigma 1 \operatorname{cf} \beta$ is a β-cardinal. In order to show $L_\beta - I \subseteq S$ we take some $x \in L_\beta - I$. We can assume without loss of generality that x is subset of a β-cardinal ρ because there is some $g \in L_\beta$ which maps x one-one into β-card(x). Let $f: \sigma 1 \operatorname{cf} \beta \to x$ be the β-recursive function which enumerates the first $\sigma 1 \operatorname{cf} \beta$ elements of x in order. We want to show that $f \in L_\beta$. For $\sigma 1 \operatorname{cf} \beta = \omega$ this is immediate since $f \upharpoonright \gamma$ is finite and therefore an element of L_ρ for every $\gamma < \sigma 1 \operatorname{cf} \beta$. For $\sigma 1 \operatorname{cf} \beta > \omega$ we know that β is a limit of limit ordinals. Therefore we can drop the assumption $\rho \leq \beta^*$ in Lemma 1.6. (Prove Lemma 1.6 as follows for these β: Take some limit ordinal λ such that $L_{\delta+1} \cup \{x\} \subseteq L_\lambda$. Consider the Skolem hull in L_λ, not in L_β.) Thus we get that $f \upharpoonright \gamma \in L_\rho$ for every $\gamma < \sigma 1 \operatorname{cf} \beta$. This implies that we can define f over some L_σ with $\sigma < \beta$. □

We can now explain the announced coherence between the notion of i-finiteness and the notion of a β-recursively enumerable set. In recursion theory one usually considers the 'universe' as a potential infinity which is in a certain sense the limit of the finite world below. Likewise one expects, that a RE set can be approximated from below by taking into consideration a larger and larger 'finite' number of steps in the associated enumeration procedure. For generalized recursive functions one can formulate this equivalently as the requirement, that every converging computation comes to an end after performing a 'finite' number of steps in the computation procedure.

It turns out that the way of counting steps in the earlier described enumeration procedure was a bit awkward. Consider therefore the following more economical way of generating L_β, which does not contain so many superfluous substeps. Fix some β-recursive cofinal strictly increasing function $q: \sigma 1 \operatorname{cf} \beta \to \beta$. We can assume without loss of generality, that there is a $\Sigma_1 L_\beta$ formula φ with the property that for every $\gamma < \sigma 1 \operatorname{cf} \beta$ $q(\gamma)$ is the minimal ordinal σ such that $L_\sigma \models \varphi(\gamma)$. Then generate L_β in $\sigma 1 \operatorname{cf} \beta$ many steps as follows. Start to build the constructible hierarchy until one reaches a level δ such that $L_\delta \models \varphi(0)$. Call this ordinal δ_0. Continue to build the L-hierarchy until one comes to the first ordinal δ such that $L_\delta \models \varphi(1)$. Call this ordinal δ_1. Etc. In this way we construct the increasing sequence of sets $\langle L_{\delta_\gamma} \mid \gamma < \sigma 1 \operatorname{cf} \beta \rangle$. Since $\delta_\gamma = q(\gamma)$ for every $\gamma < \sigma 1 \operatorname{cf} \beta$ we have that L_β is the union of these sets.

The described way of generating L_β induces a 'quick' enumeration procedure for any β-r.e. set A with $\Sigma_1 L_\beta$ definition ψ: Generate successively the sets $L_{q(0)}, \ldots, L_{q(\gamma)}, \ldots$ ($\gamma < \sigma 1 \operatorname{cf} \beta$). Enumerate at every step γ those z into A which satisfy $L_{q(\gamma)} \models \psi(z)$ and which have not already been enumerated before.

There is a certain arbitrariness concerning counting steps. But there is always a most economical way of counting steps such that the associated Kleene T-predicate "z is enumerated at step γ of the enumeration procedure e" is still β-recursive. In this sense $\sigma 1 \text{ cf } \beta$ is the minimal number of steps which is needed for the enumeration of an arbitrary β-r.e. set. The ordinal β does not have a similar significance concerning the counting of steps. One can always divide one mechanical step into many mechanical substeps.

According to Theorem 1.11 every ordinal $\gamma < \sigma 1 \text{ cf } \beta$ is an i-finite set. Furthermore I is the only recursively invariant class of β-recursive bounded subsets of L_β which contains all these ordinals. Thus we see that I is the only class of β-recursive bounded subsets of L_β, which is recursively invariant and consistent with the notion of a β-r.e. set.

We show in the following chapters that many other independent approaches to 'finite' lead to the same result. We can see immediately the equivalence of one approach that comes from the theory of admissible sets. L_β satisfies all axioms for an admissible set except possibly Δ_0-collection (we take the axiom system KP as in the book of Barwise [1]). Δ_0-collection requires that for every $a \in L_\beta$ and every Δ_0 formula φ in which b does not occur free

$$L_\beta \models \forall x \in a \, \exists y \, \varphi(x, y) \rightarrow \exists b \, \forall x \in a \, \exists y \in b \, \varphi(x, y).$$

It is tempting to call in this situation those elements a of our domain 'finite' which satisfy the collection axiom for all Δ_0 formulae φ (or equivalently for all Σ_1 formulae φ).

It is not selfevident that the collection of these 'finite' sets is 'recursive' since the definition involves several unbounded quantifiers. But it follows from the arguments in the proof of Theorem 1.11 that these 'finite' sets coincide with the i-finite sets for every L_β.

2. The admissible collapse with urelements

First we derive some basic properties of the i-finite sets.

Theorem 2.1. (a) *If K is i-finite and f is a partial β-recursive function with $K \subseteq \text{dom } f$, then $f[K]$ is i-finite.*

(b) *Every β-recursive subset of an i-finite set is i-finite.*

(c) *If K is i-finite and $f: K \rightarrow L_\beta$ is a β-recursive function with $f(x)$ i-finite for every $x \in K$, then $\bigcup \{f(x) \mid x \in K\}$ is as well i-finite.*

(d) *Assume that an i-finite set K is subset of a β-r.e. set W. If f is any β-recursive enumeration of W (i.e. $f: \sigma 1 \text{ cf } \beta \rightarrow L_\beta$ is a β-recursive function such that $f(\gamma) \subseteq f(\delta)$ for $\gamma < \delta$ and $W = \bigcup \text{range } f$), then there is some step of the enumeration where all elements of K have been enumerated (i.e. $\exists \gamma < \sigma 1 \text{ cf } \beta$ $(K \subseteq f(\gamma))$).*

In fact a β-r.e. set is i-finite if and only if it has this property.

(e) *A set $K \subseteq L_\beta$ is i-finite if and only if for every partial β-recursive function* g *there is a partial β-recursive function h such that for all $x \in L_\beta$*

$$h(x) \simeq \begin{cases} 1 & \text{if } \forall\, y \in K\, (g(x, y) \simeq 1), \\ 0 & \text{if } \exists\, y \in K\, (g(x, y) \simeq 0). \end{cases}$$

Proof. (a) Take $g \in L_\beta$ which maps some $\delta < \sigma 1 \text{ cf } \beta$ one-one onto K and use the definition of $\sigma 1 \text{ cf } \beta$ (and Theorem 1.11).

(b) Show first that the β-recursive subset is an element of L_β and then apply Theorem 1.11.

(d) It is obvious that every i-finite set K has this property. For the converse assume that K is any β-r.e. set with this property. Consider a β-recursive enumeration f of K. We get then $K = f(\gamma)$ for some $\gamma < \sigma 1 \text{ cf } \beta$. Thus $K \in L_\beta$ and if K would not be i-finite we could use a one-one map $g \in L_\beta$ from $\sigma 1 \text{ cf } \beta$ into K in order to construct an enumeration of K which does not stop before $\sigma 1 \text{ cf } \beta$.

(e) Define for an i-finite set K

$$h(x) \simeq i :\Leftrightarrow \exists\, \sigma < \beta\; L_\sigma \models [(\forall\, y \in K\, (g(x, y) \simeq 1) \wedge i = 1) \vee (\exists\, y \in K\, (g(x, y) \simeq 0) \wedge i = 0)].$$

Then h is partial β-recursive for every partial β-recursive g. On the other hand if $K \subseteq L_\beta$ is any set with the property in (e) we get immediately that K is β-recursive: consider for this

$$g(x, y) \simeq \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then choose a β-recursive function f which maps some $\delta \leq \beta$ one-one onto K (Lemma 1.9). Assume that $\delta \geq \sigma 1 \text{ cf } \beta$ (otherwise the proof is finished). Fix some $\Sigma_1 L_\beta$ formula $\psi(x)$ such that $\{x \in L_\beta \mid L_\beta \models \psi(x)\}$ is not $\Pi_1 L_\beta$. Let $q : \sigma 1 \text{ cf } \beta \to \beta$ be β-recursive and cofinal. Define $g(x, y)$ as follows for $y \in K$:

$$g(x, y) \simeq \begin{cases} 1 & \text{if } f^{-1}(y) \geq \sigma 1 \text{ cf } \beta \vee (f^{-1}(y) < \sigma 1 \text{ cf } \beta \wedge \\ & \quad L_{q(f^{-1}(y))} \models \neg \psi(x)), \\ 0 & \text{if } f^{-1}(y) < \sigma 1 \text{ cf } \beta \wedge L_{q(f^{-1}(y))} \models \psi(x). \end{cases}$$

g is partial β-recursive and the associated β-recursive function h is the characteristic function of $\{x \in L_\beta \mid L_\beta \models \psi(x)\}$. This contradiction shows that $\delta < \sigma 1 \text{ cf } \beta$.

(c) One can see easily that β-r.e. set $\bigcup \{f(x) \mid x \in K\}$ is i-finite according to the criterion in (d). □

We write in the following U for the set $L_\beta - I$. U will be the collection of urelements in the admissible set $\mathfrak{A}_\beta$.

We assume that the reader is familiar with the syntactical framework for the discussion of sets with urelements as it is presented e.g. in the book of Barwise [1] (starting on p. 9). The structure of the urelements is described in L, a first order language with equality. In our case L contains no constant symbol except equality. The structures for L will be of the form $\langle U, = \upharpoonright U\times U\rangle$. We write instead simply U because the equality symbol is always interpreted as the usual equality relation in the following. Sets with urelements are discussed in an extended language L^*. In our case L^* is a single sorted first order language like L with three additional predicate symbols for $I, \tilde{\epsilon}$ and T.

We consider in the following the structure

$$\mathfrak{A}_\beta := \langle U; I, \tilde{\epsilon}, T\rangle$$

for the language L^*. I is here again the collection of i-finite sets in L_β and $\tilde{\epsilon} := \epsilon \upharpoonright L_\beta \times I$. T is the canonical β-recursive truth predicate for $\Delta_0 L_\beta$ formulas in L_β (see e.g. Devlin [4, Lemma 8.4]).

As in the book of Barwise we use the letters x, y, z for variables in L^*. They are interpreted as ranging over $U \cup I = L_\beta$.

Lemma 2.2. *The set* $\{(x_1, \ldots, x_n) \in L_\beta^n \mid \mathfrak{A}_\beta \models \varphi(x_1, \ldots, x_n)\}$ *is β-recursive for every Δ_0 formula* $\varphi(x_1, \ldots, x_n)$ *with parameters from* L_β.

Proof. Induction on the length of φ.

(a) φ is an atomic formula.

We use here that the predicates $I, \tilde{\epsilon}, T$ are β-recursive.

(b)

$$\varphi(x_1, \ldots, x_n) \equiv \forall\, x \tilde{\epsilon} x_1\ \psi(x, x_1, \ldots, x_n),$$

$$M := \{(x, x_1, \ldots, x_n) \in L_\beta^{n+1} \mid \mathfrak{A}_\beta \models \psi(x, x_1, \ldots, x_n)\}$$

is β-recursive by the induction hypothesis. We have for any $(x_1, \ldots, x_n) \in L_\beta^n$:

$$\mathfrak{A}_\beta \models \varphi(x_1, \ldots, x_n) \Leftrightarrow x_1 \in U \vee (x_1 \in I \wedge \forall\, x \in x_1\ ((x, x_1, \ldots, x_n) \in M)).$$

This shows immediately that $\{(x_1, \ldots, x_n) \mid \mathfrak{A}_\beta \models \varphi(x_1, \ldots, x_n)\}$ is $\Pi_1 L_\beta$. In order to show that it is $\Sigma_1 L_\beta$ we observe that

$$x_1 \in I \wedge \forall\, x \in x_1\ ((x, x_1, \ldots, x_n) \in M) \Leftrightarrow$$

$$x_1 \in I \wedge \exists\, \sigma < \beta\ (x_1 \in L_\sigma \wedge L_\sigma \models \forall\, x \in x_1\ \tilde{\psi}(x, x_1, \ldots, x_n))$$

where $\tilde{\psi}$ is a $\Sigma_1 L_\beta$ definition of M.

(c) The other cases are analogous respectively trivial. □

Theorem 2.3. *$\mathfrak{A}_\beta$ is an admissible set with urelements. We have for every set* $M \subseteq L_\beta$:

$$M\, \Sigma_1\, L_\beta \Leftrightarrow M\, \Sigma_1\, \mathfrak{A}_\beta \quad \textit{and} \quad M\, \Delta_1\, L_\beta \Leftrightarrow M\, \Delta_1\, \mathfrak{A}_\beta.$$

Proof. We show first that $\mathfrak{A}_\beta$ satisfies all axioms of KPU, the theory of admissible sets with urelements (see [1]).

For any $K, H \in I$ we have in $\mathfrak{A}_\beta$ that

$$\forall x(x \bar{\in} K \leftrightarrow x \bar{\in} H) \to K = H$$

since L_β is transitive (extensionality).

The axiom of foundation is reduced to the corresponding axiom in the universe. The pairing axiom is trivial.

For the union axiom we consider a set $K \in I$. In order to prove that there is some $H \in I$ such that $\forall y \bar{\in} K\, \forall x \bar{\in} y\, (x \bar{\in} H)$ we observe that the function $f: K \to L_\beta$,

$$f(x) := \begin{cases} x & \text{if } x \in I, \\ \emptyset & \text{otherwise} \end{cases}$$

is β-recursive. Therefore $H := \bigcup \{f(x) \mid x \in K\}$ is i-finite according to Theorem 2.1(c).

The Δ_0 separation axiom requires that for any $K \in I$ and any $\Delta_0\mathfrak{A}_\beta$ formula $\varphi(x)$ the set $\{x \in K \mid \mathfrak{A}_\beta \models \varphi(x)\}$ is again i-finite. By Lemma 2.2 the set $\{x \in L_\beta \mid \mathfrak{A}_\beta \models \varphi(x)\}$ is β-recursive. Therefore the claim follows from Theorem 2.1(b).

Finally we prove the Δ_0 collection axiom. Let φ be a $\Delta_0\mathfrak{A}_\beta$ formula and K an i-finite set such that for all $x \in K$ there is some $y \in L_\beta$ with $\mathfrak{A}_\beta \models \varphi(x, y)$. We have to find an i-finite set H which contains such an y for every $x \in K$. $M := \{(x, y) \in L_\beta^2 \mid \mathfrak{A}_\beta \models \varphi(x, y)\}$ is β-recursive. We apply Σ_1 uniformization in L_β to M and get a partial β-recursive function $f \subseteq M$ with $K \subseteq \operatorname{dom} f$. $H := f[K]$ is i-finite by Theorem 2.1(a).

Thus $\mathfrak{A}_\beta$ is a model of KPU. For the rest we consider a $\Delta_0 L_\beta$ formula $\varphi(x, y)$. We show that $M := \{x \in L_\beta \mid L_\beta \models \exists y\, \varphi(x, y)\}$ is $\Sigma_1\mathfrak{A}_\beta$ definable. It is obvious that

$$x \in M \Leftrightarrow \exists\, y \in L_\beta\, \exists\, z \in L_\beta\, (z = \ulcorner \varphi(x, y) \urcorner \wedge T(z)).$$

The latter can easily be written as a $\Sigma_1\mathfrak{A}_\beta$ formula.

On the other hand it follows immediately from Lemma 2.2 that every $\Sigma_1\mathfrak{A}_\beta$ definable set is $\Sigma_1 L_\beta$ definable. $\square$

3. Infinitary languages

The notion finite is essential for ordinary logic and its model theory. A standard example is the compactness theorem: If T is a set of *finite* sentences such that every *finite* set $T_0 \subseteq T$ has a model, then T has a model.

Consider the set L_β for some limit ordinal β. We want to find out for which notion of 'finite' in L_β the compactness theorem holds. Let $L \subseteq L_\beta$ be a language as defined in Barwise [1]: L is a set of variables and symbols for relations, functions and constants together with a function which tells us the 'arity' of relation and function symbols. We assume always that L is β-recursive.

$L_{i\beta} \subseteq L_{\infty\omega}$ is defined as the set of infinitary formulas in the language L which contain conjunctions and disjunctions of i-finite sets of formulas only. $L_{i\beta}$ can be considered as a subset of L_β.

Theorem 3.1. *Assume that β is a countable limit ordinal. Let T be a $\Sigma_1 L_\beta$ set of sentences of $L_{i\beta}$. If every i-finite set $T_0 \subseteq T$ has a model, then T has a model.*

Proof. We apply the Barwise compactness theorem to the admissible collapse with urelements $\mathfrak{A}_\beta$ as defined in Section 2. $L_{i\beta}$ is the admissible fragment of $L_{\infty\omega}$ given by $\mathfrak{A}_\beta$ in the sense of Barwise [1]. By Theorem 2.3 L is $\Delta_1\mathfrak{A}_\beta$ since L is $\Delta_1 L_\beta$ and T is $\Sigma_1\mathfrak{A}_\beta$ since T is $\Sigma_1 L_\beta$. Further the sets in $\mathfrak{A}_\beta$ are exactly the i-finite sets in L_β. □

As an application of Theorem 3.1 one can show that a subset of a countable L_β is i-finite iff it is absolutely implicitly invariantly definable over L_β (see Section 5).

Theorem 3.2. *Assume that β is not admissible. Then the compactness theorem does not hold for any notion of 'finite' in L_β which satisfies*

(a) *every finite subset of L_β is 'finite' and*

(b) *every element of L_β of β-cardinality $\sigma 1 \operatorname{cf} \beta$ is 'finite' and*

(c) *every 'finite' set x is bounded (i.e. $x \subseteq L_\gamma$ for some $\gamma < \beta$).*

Proof. Assume that 'finite' satisfies (a), (b) and (c). Fix some β-recursive strictly increasing cofinal function $q: \sigma 1 \operatorname{cf} \beta \to \beta$ such that $q(0) > \sigma 1 \operatorname{cf} \beta$. We consider a language $L \subseteq L_\beta$ which contains contant symbols $\boldsymbol{\gamma} := \langle 1, \gamma \rangle$ and $\boldsymbol{q}_\gamma := \langle 1, q(\gamma) \rangle$ for $\gamma \in \sigma 1 \operatorname{cf} \beta$. Further L contains an additional constant symbol $\boldsymbol{c}$, a one place function symbol f and a two place relation symbol $=$. L is $\Delta_1 L_\beta$. Define a set of sentences in the language L as follows.

$$T := \Big\{\forall x\Big(\bigvee_{\gamma \in \sigma 1 \operatorname{cf} \beta} x = \boldsymbol{\gamma} \vee \bigvee_{\gamma \in \sigma 1 \operatorname{cf} \beta} x = f(\boldsymbol{\gamma})\Big)\Big\} \cup$$
$$\{f(\boldsymbol{\gamma}) = \boldsymbol{q}_\gamma \mid \gamma \in \sigma 1 \operatorname{cf} \beta\} \cup \{\neg \boldsymbol{c} = \boldsymbol{\gamma} \mid \gamma \in \sigma 1 \operatorname{cf} \beta\} \cup$$
$$\{\neg \boldsymbol{c} = \boldsymbol{q}_\gamma \mid \gamma \in \sigma 1 \operatorname{cf} \beta\} \cup \{\forall xyz((x = y \wedge y = z) \to x = z)\}.$$

Every formula in T is a 'finite' element of L_β by (a) and (b). Further T is $\Delta_1 L_\beta$ and every bounded $T_0 \subseteq T$ has a model. T has no model since every interpretation of $\boldsymbol{c}$ is contradictory. □

4. An equation calculus and relative recursiveness

Kleene's equation calculus is one of various formal characterizations of the recursive functions in ordinary recursion theory (see e.g. [35, §1.5]). According to Kripke [21] a similar equation calculus can be used in order to define the α-recursive functions for admissible ordinals α. Kripke adds a rule which allows to survey in a computation α-finitely many bits of information so far produced. This rule happens to be superfluous in presence of the other rules in the special case $\alpha = \omega$. Besides the approaches to α-recursion theory via definability and model theoretic invariance Kripke's equation calculus offers a way to introduce α-recursive functions via computations. One usually concentrates on this approach because computations are considered as the heart of the matter and because it is a good way to motivate the definition of relative recursiveness in α-recursion theory. Kripke's approach is discussed extensively in the early papers on metarecursion and α-recursion theory (see e.g. Kreisel and Sacks [20] or Sacks [37], a sketch is given in Shore [41]).

The following variation of the equation calculus works for every limit ordinal β. We keep Kripke's rule which allows to quantify over i-finite sets. We add some trivial initial functions. These initial functions are computable by the help of the other rules in the special case where β is admissible. It is convenient to write the equation calculus in such a way that any element of L_β may occur as argument or value of a function, not just ordinals. Since we are mainly interested in relative recursiveness we consider everything relative to a fixed set $B \subseteq L_\beta$.

A computation has the form as shown in Fig. 1.

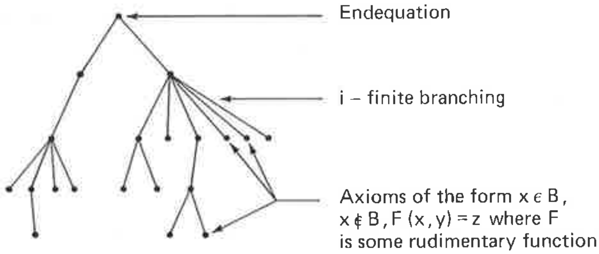

Fig. 1.

The initial functions in the equation calculus are some rudimentary functions. Jensen introduced in [17] the notion of a rudimentary function and showed that every rudimentary function can be written as the composition of nine rudimentary functions $F_0, \ldots, F_8$. Rudimentary functions are maps from the universe of sets into the universe of sets. $F_0, \ldots, F_8$ are defined as follows:

$$F_0(x, y) := \{x, y\},$$

$$F_1(x, y) := x - y,$$

$$F_2(x, y) := x \times y,$$

$$F_3(x, y) := \{\langle u, z, v\rangle \mid z \in x \wedge \langle u, v\rangle \in y\},$$

$$F_4(x, y) := \{\langle u, v, z\rangle \mid z \in x \wedge \langle u, v\rangle \in y\},$$

$$F_5(x, y) := \bigcup x,$$

$$F_6(x, y) := \operatorname{dom} x,$$

$$F_7(x, y) := \in \cap\, x^2,$$

$$F_8(x, y) := \{\{u \mid \langle z, u\rangle \in x\} \mid z \in y\}.$$

L_β is closed under all rudimentary functions. We use the restrictions of the F_i to L_β as initial functions in the equation calculus.

Observe that the rudimentary functions are a common background of β-recursion theory and Normann's set recursion [33] which generalizes recursion in objects of higher types.

The primitive symbols of the equation calculus are: function letters $f, g, h, \ldots$; variables $x, y, z, \ldots$; set constants $\boldsymbol{x}$ for every $x \in L_\beta$; function constants $\boldsymbol{F}_0, \ldots, \boldsymbol{F}_8$ (for the initial functions) and $\boldsymbol{c}_B$ (for the characteristic function of the given set B); a bounded existential quantifier $(\exists\, x \in t)$ and the equality symbol $=$.

Variables and set constants are terms. Further if f is a n-place function letter or function constant and $t_1, \ldots, t_n, t$ are terms, then $f(t_1, \ldots, t_n)$ and $(\exists\, x \in t)\, t_1$ are as well terms.

If t_1 and t_2 are terms, then $t_1 = t_2$ is an equation.

The intended meaning of the equation $(\exists\, x \in \boldsymbol{y})\, t(x) = \boldsymbol{0}$ is: y is i-finite and there is some $x \in y$ such that $t(\boldsymbol{x}) = \boldsymbol{0}$. $(\exists\, x \in \boldsymbol{y})\, t(x) = \boldsymbol{1}$ means: y is i-finite and $t(\boldsymbol{x}) = \boldsymbol{1}$ for all $x \in y$.

The axioms of the equation calculus are the equations $\boldsymbol{F}_i(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{z}$ for $x, y, z \in L_\beta$ such that $F_i(x, y) = z$, $i = 0, \ldots, 8$ and the equations

$$\boldsymbol{c}_B(\boldsymbol{x}) = \boldsymbol{0} \quad \text{for } x \in B,$$

$$\boldsymbol{c}_B(\boldsymbol{x}) = \boldsymbol{1} \quad \text{for } x \in L_\beta - B.$$

There are four computation rules:

(R1) substitute a set constant for a variable throughout an equation;

(R2) if we have computed equations $t_1 = t_2$ and $t = \boldsymbol{x}$ where t contains no variables but is not just a set constant, then we may substitute one occurrence of t in $t_1 = t_2$ by $\boldsymbol{x}$ (we call the equation $t_1 = t_2$ the major premise of this rule);

(R3) $t(\boldsymbol{x}) = \boldsymbol{0}$ for some $x \in y$ where y is i-finite $\vdash (\exists\, x \in \boldsymbol{y})\, t(x) = \boldsymbol{0}$;

(R4) $t(\boldsymbol{x}) = \boldsymbol{1}$ for every $x \in y$ where y is i-finite $\vdash (\exists\, x \in \boldsymbol{y})\, t(x) = \boldsymbol{1}$.

For a set E of equations define the set $S^{E,B}$ of all equations computable from E (and the characteristic function of B) as usual:

$S_0^{E,B}$ contains just the axioms and the equations in E. For $\delta > 0$ $S_\delta^{E,B}$ is the set of all equations which can be computed by (R1), ..., (R4) from premises in $\bigcup_{\sigma<\delta} S_\sigma^{E,B}$. $S^{E,B} := \bigcup_{\delta \in \mathrm{On}} S_\delta^{E,B}$.

Further define $S_i^{E,B} \subseteq S^{E,B}$ as the set of all equations which can be computed by an i-finite computation. In order to be able to say that a computation is i-finite assume that some coding of equations by elements of L_β is fixed. Consider a computation as a wellfounded tree where the position of every node is denoted by a finite sequence of ordinals less than β. The empty sequence is attached to the equation at the end of the computation.

Theorem 4.1. (a) *g is partial β-recursive if and only if there is a finite set E of equations such that for all $x, y \in L_\beta$*

$$g(x) \simeq y \Leftrightarrow g(\boldsymbol{x}) = \boldsymbol{y} \in S^{E,\emptyset} \Leftrightarrow g(\boldsymbol{x}) = \boldsymbol{y} \in S_i^{E,\emptyset}.$$

(b) *Assume $B \subseteq L_\beta$ and g is a partial function $L_\beta \to L_\beta$. Then there is a finite set E of equations such that for all $x, y \in L_\beta$*

$$g(x) \simeq y \Leftrightarrow g(\mathbf{x}) = \mathbf{y} \in S_i^{E,B}$$

if and only if there is a β-r.e. set W such that for all $x, y \in L_\beta$

$$g(x) \simeq y \Leftrightarrow \exists \text{ i-finite } K, H\ (\langle x, y, K, H\rangle \in W \wedge K \subseteq B \wedge H \subseteq L_\beta - B).$$

Proof. We prove at first (b). Assume E is a finite set of equations and f is a function letter such that for all $x, y \in L_\beta$

$$g(x) = y \Leftrightarrow f(\mathbf{x}) = \mathbf{y} \in S_i^{E,B}.$$

The restriction of a rudimentary function to L_β is β-recursive (Jensen [17, Corollary 1.4(b)]). Therefore the following set is β-r.e.:

$W := \{\langle x, y, K, H\rangle \mid \exists\, z \in L_\beta$ (z is an i-finite computation of $f(\mathbf{x}) = \mathbf{y}$ from equations in E and axioms where

$K = \{u \mid$ the axiom $\mathbf{c}_B(\mathbf{u}) = \mathbf{0}$ is used in the computation $z\}$

and

$H = \{u \mid$ the axiom $\mathbf{c}_B(\mathbf{u}) = \mathbf{1}$ is used in the computation $z\})\}$.

For every i-finite computation z the associated sets K and H are i-finite.

In order to prove the other direction of (b) we assume that the partial function g is defined by

$$g(x) \simeq y \Leftrightarrow \exists \text{ i-finite } K, H\ (\langle x, y, K, H\rangle \in W \wedge K \subseteq B \wedge H \subseteq L_\beta - B)$$

where $\langle x, y, K, H\rangle \in W \Leftrightarrow L_\beta \models \exists z\, \varphi(x, y, z, K, H)$ for some Δ_0 formula φ.

Assume first that g has the additional property $g(x)\downarrow \Rightarrow g(x) \neq \emptyset$.

Define auxiliary functions h_1, h_2, h_3 as follows:

$$h_1(x, u, z, v, w) = \bigcup \{y \mid y \in u \wedge L_\beta \models \varphi(x, y, z, v, w)\},$$

$$h_2(x) = \begin{cases} 0 & \text{if } \emptyset \in x, \\ 1 & \text{otherwise,} \end{cases}$$

$$h_3(x) = \begin{cases} 1 & \text{if } x \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

These functions are rudimentary according to Jensen [8].

We define a set E of six equations with function letters $h_1, \ldots, h_5, f$. The first three equations are of the form $h_i(x) = \cdots$ with a suitable composition of the $\boldsymbol{F}_0, \ldots, \boldsymbol{F}_8$ on the right side, $i = 1, 2, 3$.

(4) $\quad h_4(x) = h_2(\mathbf{c}_B(x)),$

(5) $\quad h_5(\mathbf{1}, \mathbf{1}, \mathbf{1}, y) = y,$

(6) $\quad f(x) = h_5(h_3(h_1(x, u, z, v, w)), h_3((\exists\, y \in v)\, h_4(y)),$
$\quad\quad h_3((\exists\, y \in w)\, \mathbf{c}_B(y)), h_1(x, u, z, v, w)).$

We show that for every $x, y \in L_\beta$

$$g(x) \simeq y \Leftrightarrow f(\boldsymbol{x}) = \boldsymbol{y} \in S_i^{E,B}.$$

For '$\Rightarrow$' we take $z \in L_\beta$ and i-finite $K, H \in L_\beta$ such that $L_\beta \models \varphi(x, y, z, K, H)$. Then we have $h_1(\boldsymbol{x}, \{\boldsymbol{y}\}, \boldsymbol{z}, \boldsymbol{K}, \boldsymbol{H}) = \boldsymbol{y} \in S_i^{E,B}$. Further

$$(\exists\, v \in \boldsymbol{K})\; h_4(v) = \mathbf{1} \in S_i^{E,B} \quad \text{and} \quad (\exists\, v \in \boldsymbol{H})\; \boldsymbol{c}_B(v) = \mathbf{1} \in S_i^{E,B}.$$

Together this imples

$$f(\boldsymbol{x}) = h_5(h_3(\boldsymbol{y}), h_3(\mathbf{1}), h_3(\mathbf{1}), \boldsymbol{y}) \in S_i^{E,B}.$$

By the additional assumption about g we have $y \neq \emptyset$ and therefore $f(\boldsymbol{x}) = h_5(\mathbf{1}, \mathbf{1}, \mathbf{1}, \boldsymbol{y}) \in S_i^{E,B}$. By using (5) we get from this $f(\boldsymbol{x}) = \boldsymbol{y} \in S_i^{E,B}$.

In order to show the other direction we assume that $f(\boldsymbol{x}) = \boldsymbol{y} \in S_i^{E,B}$. We can trace back the computation of this equation and get a finite sequence of equations $G_0, \ldots, G_n$ where G_0 is the endequation $f(\boldsymbol{x}) = \boldsymbol{y}$ and G_n is an axiom or an equation out of E. In this case it is necessarily eq. (6). For all $i < n$ G_i is computed from G_{i+1} by an application of rule (R1) or (R2). In the case of rule (R2) G_{i+1} is the major premise. Every G_i has necessarily the form $f(t_1) = t_2$. In course of the computation from G_n to G_0 all the auxiliary function letters and bounded quantifiers are eleminated by applications of rule (R2). Since the minor premise of these applications has the form $t = \boldsymbol{z}$ with a closed term t beginning with a bounded quantifier or a function letter we can trace back the computation of the minor premise in an analogous way. From this analysis of the computation one can see by a simple but lengthy combinatorial argument that only 'desired' equations can occur in the computation. This is shown first for the function constants $\boldsymbol{F}_0, \ldots, \boldsymbol{F}_8, \boldsymbol{c}_B$, then for the function letters h_1, h_2, h_3, then for the function letters h_4, h_5 and the terms with bounded quantifiers and finally for the principal function letter f.

It remains to be shown that we can get rid of the additional assumption $g(x)\downarrow \Rightarrow g(x) \neq \emptyset$. If g is any partial function which is defined with the help of some β-r.e. set W as above we can define the function $\tilde{g}$ with

$$\tilde{g}(x) \simeq \begin{cases} \{g(x)\} & \text{if } g(x)\downarrow, \\ \uparrow & \text{otherwise} \end{cases}$$

analogously. $\tilde{g}$ satisfies the additional assumption and is by the preceding therefore computable as desired. Since the function $x \mapsto \bigcup x$ is rudimentary this holds then as well for the function g itself. This finishes the proof of (b).

(a) is a special case of (b) where $B = \emptyset$. For any finite set E of equations one has $S^{E,\emptyset} = S_i^{E,\emptyset}$. □

We define now the notions of relative recursiveness for invariant β-recursion theory which will be studied in the rest of this paper. Analogously as in other parts of generalized recursion theory there are two possibilities. In order to compute a function g from an oracle B one can either allow any computations of equations $g(\boldsymbol{x})=\boldsymbol{y}$ which proceed according to the rules of the equation calculus (i.e. $g(\boldsymbol{x})=\boldsymbol{y}\in S^{E,B}$) or one can demand in addition that every computation tree is an element of a previously specified reservoir which does not depend on the oracle B (e.g. $g(\boldsymbol{x})=\boldsymbol{y}\in S_i^{E,B}$). In the second case one can define the reducibility as well without reference to the equation calculus according to the previous theorem.

Definition 4.2. Assume $A, B\subseteq L_\beta$.

(a) A is computable from B ($A\leqslant_c B$):$\Leftrightarrow$there is a finite set E of equations and a function letter g such that for all $x\in L_\beta$ $\chi_A(x)=j\Leftrightarrow g(\boldsymbol{x})=\boldsymbol{j}\in S^{E,B}$ (χ_A is the characteristic function of A).

(b) A is weakly i-recursive in B ($A\leqslant_{wi} B$):$\Leftrightarrow$there is a β-r.e. set W such that for all $x\in L_\beta$

$$\chi_A(x)=j\Leftrightarrow\exists \text{ i-finite } H_1, H_2\ (\langle x, j, H_1, H_2\rangle\in W\wedge H_1\subseteq B\wedge H_2\subseteq L_\beta-B).$$

(c) A is i-recursive in B ($A\leqslant_i B$):$\Leftrightarrow$there is a β-r.e. set W such that for all i-finite K

$$K\subseteq A\Leftrightarrow\exists \text{ i-finite } H_1, H_2\ (\langle K, 0, H_1, H_2\rangle\in W\wedge H_1\subseteq B\wedge H_2\subseteq L_\beta-B)$$

and

$$K\subset L_\beta-A\Leftrightarrow\exists \text{ i-finite } H_1, H_2\ (\langle K, 1, H_1, H_2\rangle\in W \\ \wedge H_1\subseteq B\wedge H_2\subseteq L_\beta-B).$$

At the first glance $\leqslant_{wi}$ seems to be the most interesting reducibility for someone who wants to admit only computations out of a fixed reservoir. But already for admissible β this reducibility is not transitive [5]. Nevertheless $\leqslant_{wi}$ remains of technical interest. In order to get a transitive reducibility one consideres instead $\leqslant_i$.

For $\beta=\omega$ all three reducibilies are the same as Turing reducibility. For $\beta=\alpha$ (α admissible) $\leqslant_c$ coincides with $\leqslant_{c\alpha}$, $\leqslant_{wi}$ with $\leqslant_{w\alpha}$ and $\leqslant_i$ with $\leqslant_\alpha$.

In α-recursion theory Kreisel has favored the first reducibility $\leqslant_c$. It has for countable α a nice model theoretic interpretation which we will extend in Section 5 to all countable β. The choice of computation rules is always to some extent arbitrary. Therefore it is satisfying to find a model theoretic interpretation which allows to state that the computation rules are in a certain sense complete. Technical interest in $\leqslant_c$ comes from the fact that the solution of Post's problem for $\leqslant_c$-requires extra work compared with the solution for $\leqslant_i$.

Of particular interest are those oracles B for which both reducibilities coincide because everything which is computable from B is already computable from B with an i-finite computation. This leads to the following definition.

Definition 4.3. $B\subseteq L_\beta$ is i-absolute if for every finite set E of equations $S^{E,B}=S_i^{E,B}$.

We show in Section 7 that nontrivial i-absolute β-r.e. sets exist for every β. The following lemma indicates how a set can be made i-absolute in a priority construction.

Lemma 4.4. *$B \subseteq L_\beta$ is i-absolute iff for every relation $R \subseteq L_\beta \times L_\beta$ with* dom R *i-finite and $\langle x, y\rangle \in R \Leftrightarrow \exists$ i-finite K, H $(\langle x, y, K, H\rangle \in W \wedge K \subseteq B \wedge H \subseteq L_\beta - B)$ for some β-r.e. W there is an i-finite function $h \subseteq R$ with* dom $h =$ dom R.

Proof. Assume B is i-absolute and consider some $R \subseteq L_\beta \times L_\beta$ as in the claim. It is obvious from Theorem 4.1(b) that there is a finite set E_1 of equations and a function letter f_1 such that for the function $g := R \times \{1\}$ we have $g(x, y) \simeq z \Leftrightarrow f_1(\mathbf{x}, \mathbf{y}) = \mathbf{z} \in S_i^{E_1, B}$. Let f be a new function symbol and define $E := E_1 \cup \{f(x) = f_1(x, y)\}$. We write K for the i-finite domain of R. For every $x \in K$ there is some y such that $\langle x, y\rangle \in R$ and therefore $f_1(\mathbf{x}, \mathbf{y}) = \mathbf{1} \in S^{E_1, B} \subseteq S^{E,B}$. Thus by (R4) $(\exists\, x \in \boldsymbol{K})$ $f(x) = \mathbf{1} \in S^{E,B}$. At this point we use the i-absoluteness of B and get $(\exists\, x \in \boldsymbol{K})$ $f(x) = \mathbf{1} \in S_i^{E,B}$.

We analyze now this i-finite computation and show that one can read off from it an i-finite uniformization function $h \subseteq R$ with dom $h =$ dom R. We go backwards in the i-finite computation of $(\exists\, x \in \boldsymbol{K})$ $f(x) = \mathbf{1}$ in the same way as in the proof of Theorem 4.1. Since E contains no equation of the form $(\exists\, x \in t) f(x) = s$ the bounded quantifier $(\exists\, x \in \boldsymbol{K})$ came in through an application of (R3) or (R4). Since we have the term $\mathbf{1}$ on the right side of the end equation this was actually an application of (R4). The premises of this application were the equations $f(\mathbf{z}) = \mathbf{1}$ for $z \in K$.

We study now the computation of $f(\mathbf{z}) = \mathbf{1} \in S_i^{E,B}$ for some $z \in K$. $f(\mathbf{z}) = \mathbf{1}$ can only be derived from $f(x) = f_1(x, y) \in E$ by (R2). The function letter f_1 can only be eliminated if it has constant arguments. Therefore the variables x and y in $f(x) = f_1(x, y)$ are first substituted by set constants $\mathbf{z}$ and $\mathbf{v}$. The minor premise of the application of (R2) where f_1 is eliminated is then the equation $f_1(\mathbf{z}, \mathbf{v}) = \mathbf{1}$. Because of the structure of E_1 we have for this v $f_1(\mathbf{z}, \mathbf{v}) = \mathbf{1} \in S_i^{E_1, B}$ and therefore $\langle z, v\rangle \in R$ by the choice of f_1, E_1.

Thus from the i-finite computation of $(\exists\, x \in \boldsymbol{K})\, f(x) = \mathbf{1}$ we can assign to every $z \in K$ the v as above with $\langle z, v\rangle \in R$ by an i-finite function h.

For the other direction one shows inductively that every equation in $S^{E,B}$ is already in $S_i^{E,B}$. □

Corollary 4.5. *Assume B is i-absolute. Then we have for all sets $A \subseteq L_\beta$*

$$A \leqslant_c B \Leftrightarrow A \leqslant_i B \Leftrightarrow A \leqslant_{wi} B.$$

Proof. One shows $A \leqslant_{wi} B \Rightarrow A \leqslant_i B$ by using the characterization of i-absolute in the preceding lemma.

$A \leqslant_c B \Rightarrow A \leqslant_i B$ follows from the definition of i-absolute. The rest is trivial. □

Remark 4.6. i-absoluteness is a property of i-degrees, not just of single sets. If B is i-absolute and $A \leq_i B$, then A is as well i-absolute (this follows immediately from Lemma 4.4).

In α-recursion theory a similar notion was introduced by Kreisel. He called a set $B \subseteq L_\beta$ *subgeneric* if $S^{E,B} = \bigcup_{\gamma<\sigma 1 \text{ cf } \beta} S_\gamma^{E,B}$ for every finite set E of equations, see Sacks [36]. There are some problems with this notion because it seems to have no equivalent definition without reference to the equation calculus (in analogy of Lemma 4.4). Thus e.g. if one wants to prove that B subgeneric together with $A \leq_i B$ implies that A is subgeneric one is drawn into painful combinatorial considerations.

Because of these problems one considers in α-recursion theory instead the notion of hyperregularity. This notion plays a key role in recent developments of the theory (see e.g. [30] and [31]).

A set $B \subseteq L_\beta$ is called *hyperregular* if for every function f such that dom f is i-finite and

$$f(x) \simeq y \Leftrightarrow \exists \text{ i-finite } H_1, H_2\ (\langle x, y, H_1, H_2\rangle \in W \wedge H_1 \subseteq B \wedge H_2 \subseteq L_\beta - B)$$

for some β-r.e. W there is some $\delta < \beta$ with $\text{Rg } f \subseteq L_\delta$.

It is obvious that if B is hyperregular and $A \leq_i B$, then A is hyperregular as well.

Lemma 4.7. *For every β and every $B \subseteq L_\beta$ we have B i-absolute $\Rightarrow$ B subgeneric $\Rightarrow$ B hyperregular.*

Proof. The first implication is obvious. For the second implication assume that B is not hyperregular and construct from the corresponding witness function a system of equations E such that some equation in $S^{E,B} - \bigcup_{\gamma<\sigma 1 \text{ cf } \beta} S_\gamma^{E,B}$ exists. $\square$

We show now that in α-recursion theory all these notions coincide for those sets which one usually studies.

Lemma 4.8. *Assume β is admissible and $B \subseteq L_\beta$ is β-r.e. or regular (i.e. $\forall \delta < \beta\ (B \cap L_\delta \in L_\beta)$). Then B i-absolute $\Leftrightarrow$ B subgeneric $\Leftrightarrow$ B hyperregular.*

Proof. Assume B is hyperregular. For B β-r.e. and β admissible this implies that B is regular. Thus we can assume that B is in addition regular. We show that the criterion for i-absolute in Lemma 4.4 is satisfied. Assume W is β-r.e. and $R \subseteq L_\beta \times L_\beta$ is a relation with dom R i-finite and

$$\langle x, y\rangle \in R \Leftrightarrow \exists \text{ i-finite } K, H\ (\langle x, y, K, H\rangle \in W \wedge K \subseteq B \wedge H \subseteq L_\beta - B).$$

Since B is regular and every initial segment of β is i-finite for admissible β we can define a function $F \subseteq R$ with dom F = dom R and

$$F(x) \simeq y \Leftrightarrow \exists \text{ i-finite } K, H\ (\langle x, y, K, H\rangle \in \tilde{W} \wedge K \subseteq B \wedge H \subseteq L_\beta - B)$$

for some β-r.e. $\tilde{W}$. Since B is hyperregular and regular this function F is in fact i-finite. □

We think that the study of i-absolute and hyperregular β-r.e. i-degrees is one of the most promising projects in invariant β-recursion theory. These two concepts do not coincide as the following example shows.

We define $\beta := \aleph_\omega + \aleph_1$ in L. Then $\sigma 1 \operatorname{cf} \beta = \aleph_1$ and $\beta^* = \aleph_\omega$. Fix some β-recursive function P which maps L_β one-one onto β^*. Then the following β-r.e. set B is hyperregular but not i-absolute: $B := \{\delta < \aleph_\omega \mid$ there is some σ and some $n \in \omega$ such that $\sigma < \aleph_n \leq \delta < \aleph_{n+1}$ and $P^{-1}(\sigma)$ is a function $h : \omega \to \aleph_\omega$ with $\forall m(\aleph_m \leq h(m) < \aleph_{m+1})$ and $\delta \leq h(n)\}$.

B is hyperregular because β has cofinality $\aleph_1$ in L (under $V = L$ B is in addition subgeneric). In order to see that B is not i-absolute consider the following relation

$$R \subseteq \omega \times L_\beta : \langle n, y\rangle \in R :\Leftrightarrow n \in \omega \wedge \aleph_n \leq y < \aleph_{n+1} \wedge y \notin B.$$

Obviously R can be defined in the required form and dom $R = \omega$. Assume $h \subseteq R$ is i-finite and dom $h = \omega$. Consider $\sigma := P(h)$ and $n \in \omega$ such tfihat $\sigma < \aleph_n$. Then $h(n) \in B$ by the definition of B and $h(n) \notin B$ because $h \subseteq R$, a contradiction.

It is quite natural that i-absoluteness and subgenericity (respecively hyperregularity) are not the same, although the assumption of admissibility obscures the difference. The former requires that everything can be computed from B with an i-finite computation, the latter requires only that everything can be computed from B with i-finite height.

5. Model theoretic invariance and infinitary logic

A sceptical mathematician might object that infinite computations are of no interest since the characteric feature of a computation is its finiteness. Further doubts may arise if one steps out into the universe of sets (Mostowski asked — perhaps rhetorically — "What is recursive in the operation of forming the union of sets?" (see [2, p. 14]). Of course a computation may behave — as we have learned — in its essential features like a finite object, although it is actually infinite. But what one considers as the essential features of finiteness may depend e.g. on the respective mathematical background. Therefore it is satisfactory that one can characterize large parts of generalized recursion theory beyond all these troublesome arguments in terms of absoluteness — or 'model theoretic invariance' as this effect was called by Kreisel [19].

Gödel [13] considers those sets $M \subseteq \omega$ which are invariantly definable in first order arithmetic A_1. This means that some formula φ in the language of A_1 exists such that in order to find out whether some natural number n belong to M or not we may take any model $\mathfrak{A}$ of A_1 and see whether $\mathfrak{A} \models \varphi(\boldsymbol{n})$ holds or not — the answer will not depend on $\mathfrak{A}$. Of course one can express this as well proof theoretically:

$$M = \{n \in \omega \mid A_1 \vdash \varphi(\boldsymbol{n})\} \quad \text{and} \quad \omega - M = \{n \in \omega \mid A_1 \vdash \neg\varphi(\boldsymbol{n})\}.$$

The sets M which are invariantly definable in this sense are exactly the recursive sets.

Kreisel noticed that one can characterize the other basic notions of recursion theory in a similar way: the recursively enumerable sets are the semi invariantly definable sets and the finite sets are the absolutely invariantly definable sets (see the generalizations in Definition 5.1). Further he suggested to consider as well invariant definability with respect to larger classes of definitions, e.g. implicit definitions which may contain an existential quantifier ranging over subsets of the model. In the unrelativized case both explict and implicit definitions lead to the same class of invariantly definable sets. Relative to a fixed set $B \subseteq L_\beta$ in general only implict definitions lead to a characterization of 'computable from B'. Kunen [22] gave a definition of 'implicitly invariantly definable' which works for countable admissible sets. He did not include the case where an additional predicate B may destroy admissibility (i.e. B is non-hyperregular). But it is well known that for countable admissible α 'implicitly invariantly definable from B over L_α' is equivalent to 'α-computable from B' (in particular stressed by Kreisel). A proof seems not to be available. We sketch a proof of the related Theorem 5.4 in order to make sure that the argument works as well in our situation (in the light of Section 2 we consider essentially admissible sets with urelements). On the way we show that for countable β 'computable from B' can as well be defined in terms of provability in infinitary logic (Theorem 5.4 (3)). In fact 'i-computable from B' can be characterized analogously (Theorem 5.5).

Finally we show that the principle of recursive invariance, which lead to the definition of i-finite in the first section, can be derived from the more general principle of model theoretic invariance.

Definition 5.1. Assume that $R, S_1, \ldots, S_k$ are subsets of L_β. Let φ be a (finite) first order formula which may contain besides $=, \tilde{\epsilon}, \tilde{R}, \tilde{S}_1, \ldots, \tilde{S}_k$ additional predicate symbols $\tilde{T}_1, \ldots, \tilde{T}_m$. φ defines R invariantly implicitly from $S_1, \ldots, S_k$ over L_β :$\Leftrightarrow$there are $T_1, \ldots, T_m \subseteq L_\beta$ such that

$$\langle L_\beta, \epsilon \restriction L_\beta \times L_\beta, R, S_1, \ldots, S_k, T_1, \ldots, T_m \rangle \models \varphi$$

and for any structure $\langle A', E', R', S'_1, \ldots, S'_k, T'_1, \ldots, T'_m \rangle$ in which φ holds and which satisfies (a), (b), (c) below we have $R = R' \cap L_\beta$; where

(a) $\langle L_\beta, \in \restriction L_\beta \times L_\beta \rangle$ is a Δ_0-elementary substructure of $\langle A', E' \rangle$;

(b) $S'_1 \cap L_\beta = S_1, \ldots, S'_k \cap L_\beta = S_k$;

(c) if $x \in L_\beta$ is i-finite and $yE'x$ for some $y \in A'$, then $y \in L_\beta$.

We say that R is invariantly implicitly definable (i.i.d.) from $S_1, \ldots, S_k$ over L_β if there is a formula φ which defines R invariantly implicitly from $S_1, \ldots, S_k$.

The relations 'R is semi invariantly implicitly definable (s.i.i.d.) from $S_1, \ldots, S_k$ over L_β' and 'R is absolutely invariantly implicitly definable (a.i.i.d.) from $S_1, \ldots, S_k$ over L_β' are defined analogously with '$R \subseteq R' \cap L_\beta$' respectively '$R = R'$' instead of '$R = R' \cap L_\beta$'.

It is obvious that for admissible β this is equivalent to Kunen's definition [22] (see also Barwise [1]).

Observe that point (a) in the definition says essentially that in our model theoretic analysis of L_β we should consider a Δ_0-formula over L_β as an atomic formula (an atomic formula is preserved in any model extension).

In the following we will not mention those fixed sets S in the list $S_1, \ldots, S_k$ which contain just a single element x. This means essentially that we consider *boldface* definitions φ where certain elements of L_β may be used as parameters (include the formula $\forall y\, \forall z\, ((\tilde{S}(y) \wedge \tilde{S}(z)) \rightarrow y = z)$ in φ in order to make sure that $S' = \{x\}$).

It is easy to see that R is i.i.d. from $S_1, \ldots, S_k$ over L_β iff R and $L_\beta - R$ are s.i.i.d. from $S_1, \ldots, S_k$ over L_β. The proof of this fact shows already why it is advisable to allow additional predicates $T_1, \ldots, T_m$ in the implicit definition.

Further for countable β a subset of L_β is a.i.i.d. over L_β iff it is i-finite. This is an application of compactness for languages with i-finite formulas (Theorem 3.1).

The following two lemmata are needed for the proof of Theorem 5.4.

Lemma 5.2. *$S^{E,B}$ is s.i.i.d. from B over L_β for every finite set of equations E and every $B \subseteq L_\beta$.*

Proof. Consider predicates $P_1, \ldots, P_6$ which are defined as follows.

$P_1(x) :\Leftrightarrow x$ is an equation of the form $\boldsymbol{F}_j(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{w}$ which is true in the standard interpretation.

$P_2(x, y, z) :\Leftrightarrow x$ is the equation $\boldsymbol{c}_B(\boldsymbol{y}) = \boldsymbol{z}$.

$P_6(x, y) :\Leftrightarrow x$ is an equation of the form $(\exists\, u \in \boldsymbol{z})\; t(u) = \mathbf{1}$ where z is i-finite and y is the set $\{t(\boldsymbol{u}) = \mathbf{1} \mid u \in z\}$.

P_3, P_4, P_5 are analogous predicates for (R1), (R2), (R3).

For every $i \in \{1, \ldots, 6\}$ we fix a $\Sigma_1 L_\beta$ definition $\exists w\, \varphi_i$ of P_i with some Δ_0 formula φ_i. The following formula defines $S^{E,B}$ semi invariantly implicitly from B over L_β:

$$\begin{aligned}\varphi :\equiv\; & \forall x (\exists w\, \varphi_1(w, x) \rightarrow \tilde{R}(x)) \\ & \wedge \forall xyz((\exists w\, \varphi_2(w, x, y, z) \wedge \tilde{B}(y) \wedge z = 0) \rightarrow \tilde{R}(x)) \wedge \forall x (x \tilde{\in} E \rightarrow \tilde{R}(x)) \\ & \wedge \cdots \wedge \forall xy((\exists w\, \varphi_6(w, x, y) \wedge \forall v \tilde{\in} y \tilde{R}(v)) \rightarrow \tilde{R}(x)). \quad \square\end{aligned}$$

In the following we consider the set $L_{i\beta}$ of i-finite formulas of a $\Delta_1 L_\beta$ language L as defined in Section 3. We take the axioms and rules for infinitary logic as in

Barwise [1, chapter III], but restricted to i-finite formulas. The symbol $\vdash$ will always refer to this notion of proof.

Lemma 5.3. *There is a finite set E of equations in the equation calculus for β-recursion theory, which contains function letters h and f, such that for any set of sentences* $T \subseteq L_{i\beta}$:

$$y \in L_{i\beta} \wedge T \vdash y \Leftrightarrow f(\mathbf{y}) = \mathbf{0} \in S^{E \cup \{h(\mathbf{x}) = \mathbf{0} \mid x \in T\}, \emptyset}$$

Proof. One has to translate the axioms and rules of infinitary logic into equations in E. This has to be done carefully so that the direction '$\Leftarrow$' can be proved as well. In view of the complexity of E it is advisable to avoid a purely syntactical proof of this direction. Therefore we proceed as follows. We define two equation systems E_1 and E_2 such that $E = E_1 \cup E_2$. E_1 consists of equations about f, h and several auxiliary function letters $h_1, \ldots, h_n$. We make sure that the function letters $f, h, h_1, \ldots, h_n$ can be interpreted in such a way by total functions $f', h', h'_1, \ldots, h'_n$ from L_β into L_β, that all equations in E_1 are satisfied in the interpretation,

$$f'(y) = \begin{cases} 0 & \text{if } y \in L_{i\beta} \wedge T \vdash y, \\ 1 & \text{otherwise,} \end{cases}$$

and all the $h'_1, \ldots, h'_n$ are β-recursive.

The equation system E_2 contains only the defining equations for $h'_1, \ldots, h'_n$ according to Theorem 4.1(a) such that

$$h'_j(x) = y \Leftrightarrow h_j(\mathbf{x}) = \mathbf{y} \in S^{E_2, \emptyset}.$$

One sees from the proof of Theorem 4.1 that further auxiliary function letters will occur in E_2 which cannot be interpreted by (partial or total) functions in such a way that all equations in E_2 are satisfied.

Consider then a computation of $f(\mathbf{y}) = \mathbf{0}$ from equations in $E \cup \{h(\mathbf{x}) = \mathbf{0} \mid x \in T\}$. At every point of the computation where some equation $h_j(\mathbf{u}) = \mathbf{v}$ is used (necessarily as minor premise in (R2) because of the structure of the equations in E_1), we cut off the computation of $h_j(\mathbf{u}) = \mathbf{v}$. Necessarily only equations from E_2 are used in this computation of $h_j(\mathbf{u}) = \mathbf{v}$ due to the structure of the equations in E_1 and E_2. Therefore we know that $h'_j(u) = v$. This implies that all the equations in the remaining torso of the computation of $f(\mathbf{y}) = \mathbf{0}$ are satisfied in the interpretation with $f', h', h'_1, \ldots, h'_n$. In particular $f'(y) = 0$ holds. This implies that $y \in L_{i\beta} \wedge T \vdash y$ because of the definition of f'.

We describe some of the equations in E_1. For the translation of the rule

$$\{\psi \rightarrow \varphi_j \mid j \in K\} \vdash \psi \rightarrow \bigwedge_{j \in K} \varphi_j \quad (K \text{ is i-finite}),$$

we use two auxiliary function letters h_1, h_2. We intend that

$$h_1'(x) = \begin{cases} \{\psi \to \varphi_j \mid j \in K\} & \text{if } x \text{ is the formula,} \\ & \psi \to \bigwedge_{j \in K} \varphi_j \text{ and } K \text{ is i-finite,} \\ \{x\} & \text{otherwise,} \end{cases}$$

and

$$h_2'(x, y) = \begin{cases} x \dot{-} y & \text{if } x, y \in \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then the following equation in E_1 takes care of the infinitary rule:

$$f(x) = h_2(\mathbf{1}, (\exists\, u \in h_1(x))\, h_2(\mathbf{1}, f(u))).$$

It is more difficult to translate the rule $\varphi \to \psi$, $\varphi \vdash \psi$ because this rule cannot be reversed. The function letter h_3 will take care of this problem. We further use h_4, h_5, h_6 where

$$h_4'(x, y) = \begin{cases} x + y & \text{if } x, y \in \omega, \\ 0 & \text{otherwise,} \end{cases}$$

$$h_5'(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise,} \end{cases}$$

$$h_6'(y, x) = \begin{cases} y \to x & \text{if } x, y \in L_{i\beta}, \\ 0 & \text{otherwise,} \end{cases}$$

We add to E_1 the equations

$$f(x) = h_2(h_5(h_4(f(h_6(y, x)), f(y)))), h_3(x, y, h_6(y, x)))$$

and

$$h_2(\mathbf{0}, h_3(x, y, z)) = \mathbf{0}.$$

We define

$$h_3'(x, y, z) = \begin{cases} 1 & \text{if } f'(x) = 0 \wedge (f'(y) = 1 \vee f'(z) = 1), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that h_3' is not β-recursive. Deviating from the treatment of the other h_j we add no equation with h_3 to E_2. The rest is analogous. ☐

Theorem 5.4. *Assume that β is a countable limit ordinal. Then for $A, B \subseteq L_\beta$ the following three relations are equivalent:*

(1) *A is computable from B;*

(2) *A is i.i.d. from B;*

(3) *for some $\Delta_1 L_\beta$ language $L \subseteq L_\beta$, which contains relation symbols $\tilde{A}$, $\tilde{B}$ and constant symbols $\boldsymbol{x}$ for every $x \in L_\beta$, and some $\Sigma_1 L_\beta$ set T of sentences in $L_{i\beta}$ the*

following holds for every $y \in L_\beta$ (*define* $\Delta_B := \{\tilde{B}(\mathbf{x}) \mid x \in B\} \cup \{\neg \tilde{B}(\mathbf{x}) \mid x \notin B\}$)

$$y \in A \Leftrightarrow T \cup \Delta_B \vdash \tilde{A}(\mathbf{y})$$

and

$$y \notin A \Leftrightarrow T \cup \Delta_B \vdash \neg \tilde{A}(\mathbf{y}).$$

Proof. (1)⇒(2): It is enough to show that A and $L_\beta - A$ are s.i.i.d. from B. Assume that f, E are such that

$$y \in A \Leftrightarrow f(\mathbf{y}) = \mathbf{0} \in S^{E,B}$$

and

$$y \notin A \Leftrightarrow f(\mathbf{y}) = \mathbf{1} \in S^{E,B}.$$

In Lemma 5.2 it was shown that $S^{E,B}$ is s.i.i.d. from B. Then $\{y \in L_\beta \mid f(\mathbf{y}) = \mathbf{0} \in S^{E,B}\}$ is as well s.i.i.d. from B (use an auxiliary predicate T for $S^{E,B}$ in the implict definition). Analogous for $y \notin A$.

(2)⇒(3): Assume that φ defines A invariantly implicitly from B over L_β according to Definition 5.1. Consider a language L which contains $=, \tilde{\epsilon}$ and all the other relation symbols in φ (but we write $\tilde{A}$ instead of $\tilde{R}$ and $\tilde{B}$ instead of $\tilde{S}_1$). Further L contains constant symbols $\mathbf{x}$ for $x \in L_\beta$.

$$T := \{\psi \mid \psi \text{ is a } \Delta_0 \text{ sentence with symbols } =, \tilde{\epsilon} \text{ and constant symbols } \mathbf{x} \text{ and } L_\beta \models \psi\} \cup \{\forall x (x \in \mathbf{y} \rightarrow \textstyle\bigvee_{u \in y} x = \mathbf{u}) \mid y \in L_\beta \text{ is i-finite}\} \cup \{\varphi\}.$$

Assume that $y \in A$. Every model of $T \cup \Delta_B$ can be construed as a structure $\mathfrak{A}' = \langle A', E', R', \ldots \rangle$ in which φ holds and which satisfies (a), (b), (c) in the definition of i.i.d. Since φ defines A invariantly implicitly from B we have $R' \cap L_\beta = A$. Therefore $\mathfrak{A}' \models \tilde{A}(\mathbf{y})$. We have thus shown that $T \cup \Delta_B \models \tilde{A}(\mathbf{y})$. The Completeness theorem for infinitary languages implies $T \cup \Delta_B \vdash \tilde{A}(\mathbf{y})$ (see [1, Exercise 4.6, p. 95] for the version which we need here). Observe that this is the only point where we use that β is countable. On the other hand if $T \cup \Delta_B \vdash \tilde{A}(\mathbf{y})$, then $y \in A$ because $\langle L_\beta, \epsilon \restriction L_\beta \times L_\beta, A, B, \ldots \rangle$ can be construed as a model for $T \cup \Delta_B$ (by the first part of the definition of i.i.d.). The proof for $y \notin A$ is analogous.

(3)⇒(1): According to Lemma 5.3 there is a finite set E of equations such that

$$f(\mathbf{z}) = \mathbf{0} \in S^{E \cup \{h(\mathbf{x}) = \mathbf{0} \mid x \in T \cup \Delta_B\}, \emptyset} \Leftrightarrow z \in L_{i\beta} \wedge T \cup \Delta_B \vdash z.$$

Extend E in two steps to E_1 and E_2. First we add equations such that

$$f(\mathbf{z}) = \mathbf{0} \in S^{E \cup \{h(\mathbf{x}) = \mathbf{0} \mid x \in T \cup \Delta_B\}, \phi} \Leftrightarrow f(\mathbf{z}) = \mathbf{0} \in S^{E_1, B}.$$

Then add a new function letter g and equations such that

$$g(\mathbf{y}) = \mathbf{0} \in S^{E_2, B} \Leftrightarrow f(\tilde{A}(\mathbf{y})) = \mathbf{0} \in S^{E_1, B}$$

and

$$g(\mathbf{y}) = \mathbf{1} \in S^{E_2, B} \Leftrightarrow f(\neg \tilde{A}(\mathbf{y})) = \mathbf{0} \in S^{E_1, B}. \quad \square$$

As a special case we get for countable β: a subset of L_β is β-recursive iff it is i.i.d. Similar arguments show that β-r.e. is equivalent to s.i.i.d.

We write $T\vdash^{i}\varphi$ if there exists an i-finite proof of φ from formulas in T.

Theorem 5.5. *Assume that β is any limit ordinal. Then for $A, B \subseteq L_\beta$ the following relations are equivalent:*

(1) $A \leqslant_i B$;

(2) *for some $\Delta_1 L_\beta$ language $L \subseteq L_\beta$, which contains relation symbols $\tilde{A}$, $\tilde{B}$ and constant symbols $\boldsymbol{x}$ for every $x \in L_\beta$, and for some $\Sigma_1 L_\beta$ set T of sentences in $L_{i\beta}$ the following holds for every i-finite set K:*

$$K \subseteq A \Leftrightarrow T \cup \Delta_B \vdash^{i} \bigwedge_{x \in K} \tilde{A}(\boldsymbol{x})$$

and

$$K \subseteq L_\beta - A \Leftrightarrow T \cup \Delta_B \vdash^{i} \bigwedge_{x \in K} \neg\tilde{A}(\boldsymbol{x}).$$

Proof. Obvious. □

Remark 5.6. (a) Although the i-finite sets coincide with the a.i.i.d. sets from Definition 5.1 one cannot use invariant definability in order to justify the definition of i-finite sets out of nothing (i-finite sets are used in Definition 5.1). But the following principle is one of those very few facts which can be derived for any conceivable notion of invariant definability: The image of an absolutely invariantly definable set under an invariantly definable function is absolutely invariantly definable. Therefore there are not many reasonable recursion theories on L_β which can be characterized in terms of invariant definability.

In order to derive the principle above one has to explicate the notion of an *invariantly definable function f*. One expects the following: If x is an element of the 'hard core' C (in our case L_β) of a collection of structures $\mathfrak{M}$, then the value $f(x)$ can be determined by inspecting the invariant definition of f over any of the structures $\mathfrak{M}$. If the graph of f (as a subset of $C \times C$) is an invariantly definable set it is nevertheless possible that besides $(x, f(x))$ some pair (x, z) with $z \notin C$ satisfies the invariant definition of the graph over a certain $\mathfrak{M}$. Then it is impossible to determine the correct value over $\mathfrak{M}$. Therefore one has to demand in addition that the invariant definition of the graph of f defines a function over any of the structures $\mathfrak{M}$ in question. This is no restriction for any notion of implicit invariant definability. If $\varphi(\tilde{R})$ defines i.i. the graph of f as a set, then $\varphi(\tilde{R}) \wedge \forall xyz(\tilde{R}(x, y) \wedge \tilde{R}(x, z) \rightarrow y = z)$ defines f i.i. as a function.

It is then trivial to derive the claimed principle.

(b) Formulas φ_1 and φ_2 which define according to Definition 5.1 s.i.i.d. sets $B_1, B_2 \subseteq L_\beta$ such that B_1 is not i.i.d. from B_2 and B_2 is not i.i.d. from B_1 might look a bit unpleasant. Nevertheless such formulas exist (at least for countable β) as it is proved in Theorem 7.1 in terms of computations. Thus it is the combination of both aspects which makes the theory interesting.

6. More models for axiomatic computation theories

We consider axioms for computation theories which go back to Moschovakis [32] (he was stimulated by Kleene's schemes for recursion in higher types [18]). These axioms were extensively studied by Fenstad [7]. We use the definitions of his survey paper [6].

A set Θ of triples (a, σ, z) is considered which satisfies certain closure and uniformity conditions. The intention of $(a, \sigma, z) \in \Theta$ is: $\{a\}(\sigma) \simeq z$, where a codes some computing device. Further there is a well-founded transitive relation $<$ on Θ. The axioms demand that $<$ behaves like 'is subcomputation of'.

A function f is called Θ-computable if for some a:

$$f(\sigma) \simeq z \Leftrightarrow (a, \sigma, z) \in \Theta.$$

A set is called Θ-semicomputable if it is the domain of some Θ-computable function. A set is called Θ-finite if one can Θ-computable quantify over it. A nice feature of this definition is the possibility to characterize those computation theories which generalize recursion in normal objects of higher types (Spector theories): these are the computation theories where the whole domain is Θ-finite.

A computation theory is called p-normal if it allows some kind of stage comparison. It is called s-normal if for any $(a, \sigma, z) \in \Theta$ the set of 'subcomputations' $\{(a', \sigma', z') \mid (a', \sigma', z') < (a, \sigma, z)\}$ is Θ-finite in a uniform way (this is related to the coherence requirement in Section 1).

Theorem 6.1. *Assume that β is a limit ordinal. There is a p-normal and s-normal computation theory $\langle \Theta, < \rangle$ such that the Θ-computable functions are exactly the β-recursive functions, the Θ-semicomputable sets are exactly the β-r.e. sets and the Θ-finite sets are exactly the i-finite sets.*

Proof. Straightforward. See the related characterization of i-finite sets in Theorem 2.1(e).

Remark 6.2. (a) Originally Moschovakis considered instead of a subcomputation relation $<$ a map from Θ into the ordinals, which gives the 'length' of a computation. Usually both versions can be used. Theorem 6.1 holds in general only for the refined version with 'subcomputations' due to Fenstad.

(b) The Θ-finite sets are invariant under Θ-computable permutations of the universe for all computation theories.

7. Post's problem

We consider here the notions of relative recursiveness which were defined in Section 4. We concentrate on i-degrees, which are the equivalence classes with respect to the relation $\leqslant_i$. One verifies immediately that for all β there is a smallest i-degree $\mathbf{0}_i$ (the i-degree of the empty set) and a largest β-r.e. i-degree $\mathbf{0}_i'$ (the i-degree of an universal β-r.e. set). A set is β-recursive if and only if it is in the degree $\mathbf{0}_i$. Further $\mathbf{0}_i <_i \mathbf{0}_i'$.

Since i-degrees coincide with β-degrees for admissible β, new questions arise only for inadmissible β. Further for all β the i-degrees coincide with the $\mathfrak{A}_\beta$-degrees in the admissible collapse $\mathfrak{A}_\beta$. Now if β is weakly inadmissible (i.e. $\beta^* \leqslant \sigma 1 \text{ cf } \beta < \beta$) the admissible collapse has a particularly nice representation according to [27]. One can write it as an admissible structure $\langle L_{\sigma 1 \text{cf} \beta}, \varepsilon, T \rangle$ where T is a regular predicate over $L_{\sigma 1 \text{cf} \beta}$ which preserves the fine structure of $L_{\sigma 1 \text{cf} \beta}$. Therefore constructions of α-recursion theory can be extended to the admissible collapse of a weakly inadmissible β.

The interesting open questions arise in the case where β is strongly inadmissible (i.e. $\sigma 1 \text{ cf } \beta < \beta^*$). For these β the i-degrees still coincide with the degrees in the admissible collapse $\mathfrak{A}_\beta$. But $\mathfrak{A}_\beta$ is in this case a very fat admissible set, where no construction of α-recursion theory succeeds (Stoltenberg-Hansen has shown that these are exactly those β where $\mathfrak{A}_\beta$ is not resolvable, see [7, Theorem 6.3.14]). In fact one cannot expect that all results from α-recursion theory can be extended to all fat admissible sets because Harrington has constructed such a set where $\mathbf{0}$ and $\mathbf{0}'$ are the only Σ_1-degrees [14].

Thus for strongly inadmissible β the admissible collapse $\mathfrak{A}_\beta$ only supplies 'soft' results about β like the Barwise compactness theorem in Section 3. Concerning 'hard' results the information flows in the other direction. Although $\mathfrak{A}_\beta$ may be an enormously fat admissible set (consider e.g. $\beta = \aleph_1 + \omega$) it has still got some regularity which comes from the fine structure of L_β. Therefore one can in fact solve Post's problem for these fat admissible sets $\mathfrak{A}_\beta$ according to the following theorem.

Theorem 7.1. *Assume β is any limit ordinal. Then there are β-r.e. sets A, B of incomparable i-degree. We can make A, B in addition i-absolute.*

Proof. By our previous remarks the solution of Post's problem in α-recursion theory by Sacks and Simpson [38] covers as well the case where β is weakly inadmissible. For strongly inadmissible β with β^* β-recursively regular Friedman [10] has constructed β-r.e. sets A, B which are incomparable w.r.t. to $\leqslant_{w\beta}$ and therefore as well incomparable w.r.t. $\leqslant_i$ (observe that we don't get this if A, B are just incomparable w.r.t. $\leqslant_\beta$). In all these cases one can make A, B in addition i-absolute by adding negative requirements as in the construction below.

We assume now for the rest of the proof that β is strongly inadmissible and that β^* is not β-recursively regular (i.e. there is a β-recursive function which maps some $\delta < \beta^*$ cofinally into β^*). Obviously β^* is in this case a limit of β-cardinals. For any $\delta < \beta^*$ we write δ^+ for the next β-cardinal after δ.

For all β-cardinals $\zeta < \beta^*$ we proceed between ζ and ζ^+ similarly as Friedman proceeds between 0 and β^* in the case where β^* is β-recursively regular. It just remains to be shown that these different segments of the construction do not interfere with each other in a serious way.

Jensen [17] has shown that the combinatorial principle $\diamond$ holds for all regular cardinals in L. Friedman [10] has introduced effective versions of $\diamond$ for β-recursively regular β-cardinals. If β^* is not β-recursively regular one can piece together the $\diamond$-sequences for the β-recursively regular β-cardinals below β^*. Thus we define the $\diamond$-sequence $\langle S_\delta \mid \delta < \beta^* \rangle \in L_\beta$ by:

$$S_\delta := \begin{cases} \phi & \text{if } \delta \text{ is a } \beta\text{-cardinal,} \\ 2^\delta \cap L_{\hat{\delta}}, & \text{where } \hat{\delta} := \mu\gamma \geq \delta(L_\gamma \models [\delta \text{ is not cardinal}]) \text{ otherwise.} \end{cases}$$

Then for every $\delta < \beta^*$ the β-cardinality of S_δ is less or equal to the β-cardinality of δ.

Let p be an element of L_β. For a β-cardinal $\rho < \beta^*$ we define

$$C_{p,\rho} := \{\delta \mid \rho < \delta < \rho^+ \wedge h_1[(\delta \cup \{p\}) \times \omega] \cap \rho^+ = \delta\}$$

where h_1 is a parameter free Σ_1-skolem function for L_β. It is then easy to show that $C_{p,\rho}$ is closed and unbounded in ρ^+. Further if $\delta \in C_{p,\rho}$ and the set has a Σ_1 definition over L_β with parameter p, then $W \cap \delta \in S_\delta$ (see Friedman [10] for proofs).

Fix a β-recursive function P which maps L_β one-one onto β^*. Further fix a β-recursive strictly increasing cofinal function $q: \sigma 1 \operatorname{cf} \beta \to \beta$.

The construction of A and B proceeds in $\sigma 1 \operatorname{cf} \beta$ many steps. At every step $\gamma < \sigma 1 \operatorname{cf} \beta$ we consider every stage $\delta < \beta^*$ which is not a β-cardinal. At stage $\delta < \beta^*$ we consider all requirements $R^A_{e,X}$, $R^B_{e,X}$, $N^A_{e,X}$, $N^B_{e,X}$ with $e < \delta$ and $X \in S_\delta$. The β-cardinality of these requirements equals the β-cardinality of δ. We assume that for every stage δ a well ordering of the requirements at stage δ has been assigned in a β-recursive way. If S_1, S_2 are two requirements, then S_1 has higher priority than S_2 if either S_1 is a requirement at lower stage than S_2 or S_1 and S_2 are at the same stage and S_1 precedes S_2 in the well ordering of requirements at that stage.

$R^A_{e,X}$ tries to prevent that for all $x \in \beta^*$

$$x \in \beta^* - A \leftrightarrow \exists \quad \text{i-finite} \quad H\ (\langle x, H \rangle \in W_{P^{-1}(e)} \wedge H \subseteq \beta^* - B).$$

$N^A_{e,X}$ tries to make sure that if $(P^{-1}(e))_0$ is a function which maps some $\gamma < \sigma 1 \operatorname{cf} \beta$ one-one onto dom R, where

$$R := \{\langle x, y \rangle \mid \exists \quad \text{i-finite} \quad H\ (\langle x, y, H \rangle \in W_{(P^{-1}(e))_1} \wedge H \subseteq \beta^* - A)\},$$

then there is an i-finite function $h \subseteq R$ with dom $h = \operatorname{dom} R$.

$R^A_{e,X}$ uses X as a guess at $B \cap \delta$. $N^A_{e,X}$ uses X as a guess at $A \cap \delta$.

Step γ, *Stage* δ: Assume $R^A_{e,X}$ is the next requirement to be considered, $L_{q(\gamma)} \models \exists y (P(y) = e)$ and nothing has been done for $R^A_{e,X}$ at stage δ at a previous step. Then we check whether there is some pair $\langle x, H \rangle$ with $\delta < x < \delta^+$ and H

i-finite such that x was not restrained from A for a requirement at stage $\leq\delta$, $\langle x, H\rangle \in W_{P^{-1}(e)}^{L_{q(\gamma)}}$, $H\cap X=\emptyset$ and no element of H is already in B. If it exists, we take the least such pair, enumerate x in A and restrain all elements of $H-\delta$ from B for $R^A_{e,X}$ at stage δ.

Assume $N^A_{e,X}$ is the next requirement to be considered and $L_{q(\gamma)}\models\exists y(P(y)=e)$. We do only something for $N^A_{e,X}$ if $(P^{-1}(e))_0$ is a function f with $\operatorname{dom} f\in\sigma 1 \operatorname{cf}\beta$ and if there are ordinals $v\in\operatorname{dom} f$ such that for some i-finite H and some y $\langle f(v), y, H\rangle\in W_{(P^{-1}(e))_1}^{L_{q(\gamma)}}$, $H\cap X=\emptyset$ and no element of H is already in A. Then for all v where not already some computation was preserved for $N^A_{e,X}$ at stage δ we preserve now a computation by choosing the tripel $\langle f(v), y, H\rangle$ above minimal and restraining $H-\delta$ from A for $N^A_{e,X}$ at stage δ.

The requirements $R^B_{e,X}$, $N^B_{e,X}$ are treated analogously. End of the construction.

For every requirement R on stage δ at most one element x with $\delta<x<\delta^+$ is enumerated in A or B during the construction and at most the elements of an i-finite set K are altogether restrained from A or B during the construction. The latter follows for $R\equiv N^A_{e,X}$ from the fact that the set of steps γ where new computations are preserved for $N^A_{e,X}$ at stage δ is bounded below $\sigma 1 \operatorname{cf}\beta$ (we use here that $\sigma 1 \operatorname{cf}\beta\leq\beta^*$). Assume then for a contradiction that for all $x\in\beta^*$

$$x\in\beta^*-A\leftrightarrow\exists \text{ i-finite } H\,(\langle x, H\rangle\in W_{P^{-1}(e)}\wedge H\subseteq\beta^*-B).$$

Consider some stage $\delta>e$ such that $\delta\in C_{p,\rho}$ for some β-cardinal ρ with $\sigma 1 \operatorname{cf}\beta<\rho<\beta^*$, where p is the parameter of the construction. Then $B\cap\delta\in S_\delta$ and no element y with $\delta<y<\rho^+$ is ever enumerated or restrained for a requirement at some stage $<\delta$.

$C_{p,\rho}$ is unbounded in ρ^+ and no element of $C_{p,\rho}$ is ever enumerated in A (because at stage δ only elements $x>\delta$ are enumerated). Therefore β^*-A is unbounded in ρ^+. Consider some $x\in\beta^*-A$ with $\delta<x<\rho^+$ such that x is never restrained from A for a requirement at stage δ together with some i-finite H with $\langle x, H\rangle\in W_{p^{-1}(e)}$ and $H\subseteq\beta^*-B$. Then for the requirement $R^A_{e,B\cap\delta}$ at stage δ we can always do something from some step γ_0 on as witnessed by the pair $\langle x, H\rangle$. Therefore there is a step γ where for $R^A_{e,B\cap\delta}$ at stage δ a pair $\langle x', H'\rangle$ is chosen, x' is enumerated in A and all elements of $H'-\delta$ are restrained from B. Then $\langle x', H'\rangle\in W_{P^{-1}(e)}$ and $H'\cap B\cap\delta=\emptyset$. Because of the choice of δ no element of $H'-\delta$ is afterwards enumerated in B for a requirement at a stage less than δ (we use here that according to the construction only elements less than σ^+ are enumerated for requirements at stage σ). Further by construction no element of $H'-\delta$ is afterwards enumerated in B for a requirement at a stage $\geq\delta$. Thus $H'\subseteq\beta^*-B$ and $x'\in A$, a contradiction.

Finally assume that for some $y_0\in L_\beta$

$$R:=\{\langle x, y\rangle\mid\exists \text{ i-finite } H\,(\langle x, y, H\rangle\in W_{y_0}\wedge H\subseteq\beta^*-A)\}$$

has an i-finite domain. Let f be an i-finite function which maps some $\gamma_0<\sigma 1 \operatorname{cf}\beta$

one-one onto dom R. Define $e := P(\langle f, y_0\rangle)$. Consider as before some stage $\delta > e$ such that $\delta \in C_{p,\rho}$ for some β-cardinal ρ with $\sigma 1 \text{ cf } \beta < \rho < \beta^*$. Then no element y which is restrained for $N^A_{e,A\cap\delta}$ from A will ever come into A (same argument as before). Thus for every $v < \gamma_0$ there is exactly one computation $\langle f(v), y_v, H_v\rangle$ permanently preserved for $N^A_{e,A\cap\delta}$ at stage δ. Further there is a step $\gamma_1 < \sigma 1 \text{ cf } \beta$ at which for every $v < \gamma_0$ such a computation $\langle f(v), y_v, H_v\rangle$ has been preserved. Therefore the function h from dom R into L_β which maps $f(v)$ on y_v for every $v < \gamma_0$ is an element of L_β and thus i-finite. Obviously we have dom h = dom R and $h \subseteq R$.

This finishes the proof of Theorem 7.1.

Observe that if one does not intend to make A, B i-absolute a simpler guessing sequence $\langle S_\delta, \delta < \beta^*\rangle$ is sufficient. Since in this case one has to preserve for every requirement at most one computation one can simply take for S_δ all i-finite subsets of δ. It is tempting to think that these S_δ are as well sufficient for the requirements $N_{e,X}$ But for the considered relations R one cannot a priori fix an i-finite set $\bar{H}$ such that only computations with neighborhoods $H \subseteq \bar{H}$ have to be considered (although we can do this after we know that A, B are i-absolute).

8. Comparison with Friedman-Sacks' β-recursion theory

Friedman and Sacks [8] have introduced a different recursion theory on limit ordinals β. They define β-r.e. and β-recursive in the same way but use a different notion of 'finite':

$$x \subseteq L_\beta \text{ is } \beta\text{-finite } :\Leftrightarrow x \in L_\beta.$$

This β-recursion theory has been studied in several papers by Friedman [9, 10, 11, 12], Homer [15] Stoltenberg-Hansen [42] and the author [26, 27, 29, 31]. We call this theory FS β-recursion theory.

FS β-recursion theory is closely connected to the study of Σ_2 sets in α-recursion theory. Problems about $\Sigma_2 L_\alpha$ sets like the existence of sets of minimal α-degree for all admissible α have remained unsolved for a long time. The study of $\Sigma_2 L_\alpha$ sets is equivalent to the study of subsets of L_α which are Σ_1 definable over $\langle L_\alpha, \epsilon, C\rangle$, where C is a complete regular α-r.e. set. This structure has basically the same fine structure as an initialsegment of L but it is in general inadmissible (if L_α is not Σ_2 admissible). Therefore from the interest in Σ_1 sets over $\langle L_\alpha, \epsilon, C\rangle$ one is naturally lead into a systematic study of Σ_1 sets over L_β for inadmissible β.

The corresponding notion of 'finite' suggests itself from the paradigm $\langle L_\alpha, \epsilon, C\rangle$. Since one is still interested in α-degrees (where α-finite sets are used as 'finite' sets), one calls a set 'finite' iff it is an element of the considered universe. If one analyzes this step in terms of invariance under permutations of the universe one arrives at the following observation. Even if one studies $\Sigma_2 L_\alpha$ sets "as if they were r.e." by considering them as Σ_1 sets over $\langle L_\alpha, \in, C\rangle$ one is still interested in results about α-recursion theory. Thus the characteristic invariance group is still the set of all α-recursive permutations of L_α — not the set of all $\Sigma_2 L_\alpha$-permutations of L_α.

But as soon as one starts a new recursion theory where 'r.e.' is Σ_1 over $\langle L_\alpha, \in, C\rangle$ it makes sense to adopt as the characteristic invariance group for this theory the group of all 'recursive' (i.e. Δ_1 $\langle L_\alpha, \in, C\rangle$) permutation of the universe L_α. It is clear that the first point of view leads to FS β-recursion theory and the second to invariant β-recursion theory.

FS β-recursion has been very successful concerning the solution of open problems about $\Sigma_2 L_\alpha$ sets (see e.g. the existence of incomparable α-degrees above $0'$ [12] or the characterization of the jump of α-r.e. degrees [30, 31]).

If one studies FS β-recursion theory for its own sake several strange effects arise. There are β-recursive sets of nonzero β-degree, there are β-finite subsets of β-r.e. sets W which at no point of the enumeration of W are completely enumerated and there are β-r.e. sets which are β-recursively isomorphic but which have a different β-degree. Further the definition of 'β-recursive in' is lifted verbatim from α-recursion theory although for inadmissible β there is no computation calculus with β-finite computations in the background which justifies this definition. Therefore central points of α-recursion theory (e.g. absoluteness effects like hyperregularity) become meaningless in FS β-recursion theory.

We expect that one gets in invariant β-recursion theory more uniform results. Many considerations in FS β-recursion theory split into cases because of the lacking invariance (e.g. the β-degree of a set depends on the chosen representation, in $\beta := \omega + \omega$ the β-degree of a set $A \subseteq \omega$ is in general different from the β-degree of the set $\{\omega + n \mid n \in A\}$). Constructions in invariant β-recursion theory keep an unmistakable recursion theoretic flavor because in this theory a computation from a β-r.e. set behaves as in classical recursion theory. This is due to the fact that every i-finite subset of a β-r.e. set W is completely enumerated at some point of the enumeration of W. New strategies are only needed because an enormous number of requirements have to be satisfied in a very short time.

We consider invariant β-recursion theory as an attempt to capture the fascinating hard construction problems which arise if one drops the assumption of admissibility and to present at the same time a sound conceptual framework.

We show in the following two theorems that one can easily recover large parts of the structure of β-degrees inside the structure of i-degrees.

Observe that in general $A \leq_\beta B$ does not imply $A \leq_i B$ and $A \leq_i B$ does not imply $A \leq_\beta B$.

Theorem 8.1. *For every β one can embed the β-recursive β-degrees into the β-r.e. i-degrees (both considered as partial orders).*

Proof. For $A \subseteq L_\beta$ define

$$A_i := \{\langle K, 0\rangle \mid K \in L_\beta \wedge K \cap A \neq \emptyset\}$$
$$\cup \{\langle K, 1\rangle \mid K \in L_\beta \wedge K \cap (L_\beta - A) \neq \emptyset\}.$$

A_i if β-r.e. if A is β-recursive.

One can translate i-finite neighborhood conditions $H \subseteq L_\beta - A_i$ into β-finite neighborhood conditions $K_1 \subseteq A$, $K_2 \subseteq L_\beta - A$:

$$H \subseteq L_\beta - A_i \Leftrightarrow$$
$$\exists K_1, K_2 \in L_\beta (K_1 = \bigcup\{K \mid \langle K, 1\rangle \in H\}$$
$$\wedge K_2 = \bigcup\{K \mid \langle K, 0\rangle \in H\} \wedge K_1 \subseteq A \wedge K_2 \subseteq L_\beta - A).$$

Therefore $A \leqslant_\beta B$ implies $A_i \leqslant_i B_i$ for β-recursive A, B: $H \subseteq L_\beta - A_i$ for i-finite H is reduced to β-finite neighborhood conditions $K_1 \subseteq A$, $K_2 \subseteq L_\beta - A$. These are reduced to β-finite neighborhood conditions $\tilde{K}_1 \subseteq B$, $\tilde{K}_2 \subseteq L_\beta - B$ (because $A \leqslant_\beta B$) and this is equivalent to $\langle \tilde{K}_1, 1\rangle \in L_\beta - B_i$, $\langle \tilde{K}_2, 0\rangle \in L_\beta - B_i$.

Since A_i is β-r.e. we need not consider i-finite neighborhood conditions $H \subseteq A_i$.

In order to show $A_i \leqslant_i B_i \Rightarrow A \leqslant_\beta B$ we use the same translations.

In order to get a degree embedding E we define for a β-recursive β-degree $\boldsymbol{a}_\beta$ $E(\boldsymbol{a}_\beta)$ as the i-degree of A_i for some β-recursive $A \in \boldsymbol{a}_\beta$. □

It is easy to check that the degree embedding E from the previous proof always maps the least β-degree $\mathbf{0}_\beta$ on the least i-degree $\mathbf{0}_i$. Further it maps for all inadmissible β the largest β-recursive β-degree $\mathbf{0}_\beta^{1/2}$ on the largest β-r.e. i-degree $\mathbf{0}'$.

We have shown in [27] that for weakly inadmissible β one can embed the β-recursive β-degrees one-one *onto* the β-r.e. i-degrees (observe that the degrees in the admissible collapse of β are exactly the i-degrees). Such an isomorphism is not possible for all inadmissible β. Incomparable β-r.e. i-degrees exist for all β (Theorem 7.1) but there are strongly inadmissible β without incomparable β-recursive β-degrees (Friedman [11]). It will be interesting to see for which β an isomorphism exists.

Theorem 8.2. *Assume β is strongly inadmissible and β^* is Σ_2-regular (i.e. no $\Sigma_2 L_\beta$ function maps some $\delta < \beta^*$ cofinally into β^*). Then one can embed the β-r.e. β-degrees into those i-degrees $\boldsymbol{a}_i$ which are r.e. in some β-r.e. i-degree $\boldsymbol{b}_i \leqslant_i \boldsymbol{a}_i$.*

Proof. We use the same embedding as in the proof of the previous theorem: for A β-r.e. we define $E(\beta\text{-deg}(A))$ as the i-degree of A_i. We have $\langle K, 1\rangle \in A_i \Leftrightarrow \exists x \in K\ (x \notin A)$. Therefore A_i is r.e. in A (more exactly one might say 'i-r.e. in A' since only i-finite conditions about A are used). Further $A \leqslant_i A_i$ (trivial). Therefore $\boldsymbol{a}_i := \text{i-deg}(A_i)$ is as required with $\boldsymbol{b}_i := \text{i-deg}(A)$.

It remains to show that for β-r.e. sets A, B: $A \leq_\beta B \Leftrightarrow A_i \leq_i B_i$. This follows from the following consideration which enables us to translate for β-r.e. A i-finite conditions about A_i into β-finite conditions about A.

Assume H is i-finite and contains only elements of the form $\langle K, 1\rangle$ with $K \in L_\beta$. Then $H \subseteq A_i$ means that $\forall \langle K, 1\rangle \in H \, \exists x \in K \, (x \notin A)$. We show that in this case there exists in fact a β-finite set $\tilde{K} \subseteq L_\beta - A$ such that

$$\forall \langle K, 1\rangle \in H \, \exists x \in K \, (x \in \tilde{K}).$$

Fix a β-recursive function P which maps L_β one-one onto β^*. Assume H is i-finite and $\forall \langle K, 1\rangle \in H \, \exists x \in K \, (x \notin A)$. Then the following relation R has dom $R = H$:

$$R := \{\langle\langle K, 1\rangle, y\rangle \mid \langle K, 1\rangle \in H \wedge P^{-1}(y) \in K \wedge P^{-1}(y) \notin A\}.$$

R is $\Sigma_2 L_\beta$ and therefore by Jensen's uniformization theorem [17] there is a $\Sigma_2 L_\beta$ function $F \subseteq R$ with dom $F =$ dom R. Since H is i-finite and β^* is Σ_2-regular there is some $\delta < \beta^*$ with Rg $F \subseteq \delta$. Then $K_0 := P[A] \cap \delta \in L_\beta$. Since we can't be sure that $P^{-1}[\delta - K_0] \in L_\beta$ we have to shrink it a little more. By using Σ_1 uniformization we get a $\Sigma_1 L_\beta$ function F' which maps every $\langle K, 1\rangle \in W$ on some $y \in \delta - K_0$ with $P^{-1}(y) \in K$. This function F' is then in fact i-finite and therefore $\tilde{K} := P^{-1}[\text{Rg } F']$ is as well i-finite, in particular an element of L_β.

We have shown in [31] that for some weakly inadmissible β the β-r.e. β-degrees are isomorphic to the i-degrees of the preceding theorem.

9. The lattice of β-r.e. sets

We are looking for a notion of 'finite' which is adequate for the study of the lattice of β-r.e. sets for all β. The analogous step from ω to α was done by Lerman [24] (see also his survey [25]). Lerman points out that the α-finite sets are not an ideal if $\alpha^* < \alpha$. On the other hand he shows that the α^*-finite sets (these are those α-finite sets which have α-cardinality less than α^*) capture most of the properties which are characteristic for finite sets in the lattice of r.e. sets.

We define below for every limit ordinal β the notion of a l-finite set (l for lattice). We suggest to consider this notion as the generalization of finite in the lattice of β-r.e. sets. For admissible β this notion coincides with Lerman's α^*-finite set.

Lemma 9.1. *Assume β is any limit ordinal and $M \subseteq L_\beta$. Then the following are equivalent*:

(a) *there is some $\delta < \beta^*$ and a β-recursive permutation f of L_β such that $M = f[\delta]$*;

(b) *M is β-r.e. and every β-r.e. subset of M is β-recursive*;

(c) *M is β-recursive and there is no β-recursive function which maps L_β one-one into M;*

(d) *M is β-recursive and there is some $\delta < \beta^*$ and some β-recursive function which maps δ one-one onto M.*

Proof. (d) $\Rightarrow$ (a): Assume M is β-recursive and g is a β-recursive function which maps δ one-one onto M. $L_\beta - M$ is β-r.e. and therefore according to Lemma 1.9 there is some $\delta_1 \leq \max(\beta^*, \sigma 1 \text{ cf } \beta)$ and a β-recursive function h_1 which maps δ_1 one-one onto $L_\beta - M$. We have $\delta_1 \geq \beta_1^*$ because otherwise one could combine h_1^{-1} and g^{-1} in order to project L_β β-recursively into some ordinal less than β^*. Further we have $\delta_1 \geq \sigma 1 \text{ cf } \beta$, because otherwise $\beta^* < \sigma 1 \text{ cf } \beta$ and both g and h_1 have bounded range in L_β. Thus $\delta_1 = \max(\beta^*, \sigma 1 \text{ cf } \beta)$.

One shows analogously that there is a β-recursive function h_2 which maps $\max(\beta^*, \sigma 1 \text{ cf } \beta)$ one-one onto $L_\beta - \delta$. $h_1 \circ h_2^{-1}$ can then be used in order to extend $g : \delta \to M$ to a β-recursive permutation of L_β.

(a) $\Rightarrow$ (d) is trivial.

One verifies easily that (d) is equivalent to (b) and (c) by using again Lemma 1.9 and the basic properties of β^*.

Definition 9.2. A set $M \subseteq L_\beta$ is called *l-finite* if there is some $\delta < \beta^*$ and a β-recursive permutation f of L_β such that $M = f[\delta]$.

It is obvious that i-finite sets can be defined in the same fashion with $\sigma 1 \text{ cf } \beta$ instead of β^*. Thus for every β the ordinals $\sigma 1 \text{ cf } \beta$ and β^* are the two numbers which are characteristic for the two basic aspects of finiteness in β-recursion theory. In the context of computations one arrives at i-finite sets, in the context of the lattice of β-r.e. sets one arrives at l-finite sets. For the former aspect boundedness is an essential part of 'finite'. This is different in the lattice of r.e. sets where finite sets play the role of 'sets of measure zero'. Thus in principle they might even be unbounded as long as they are thin enough. This phenomenon did not appear in α-recursive theory because there are no thin α-recursive cofinal sets. But if $\sigma 1 \text{ cf } \beta < \beta^*$ there exist very thin β-recursive cofinal sets.

The aspect of l-finite sets as 'sets of measure zero' is described in (c) of Lemma 9.1. In part (b) of this lemma it is verified that l-finite sets possess another property which is characteristic for finite sets in the lattice of r.e. sets: the induced lattice on a l-finite set is trivial (a Boolean algebra). Further from (b) one sees immediately that the l-finite set form a definable ideal in the lattice of β-r.e. sets.

We study in this paper only one other lattice theoretic concept: simple sets.

In order to give a correct definition of simple sets in β-recursion theory we consider $\mathfrak{E}^*(\beta)$ — the quotient lattice of the lattice of β-r.e. sets obtained upon factoring by the ideal of l-finite sets. The elements of $\mathfrak{E}^*(\beta)$ are equivalence classes with respect to the congruence relation:

$$U \approx V :\Leftrightarrow \exists \text{ l-finite } I\, [U \subseteq V \cup I \wedge V \subseteq U \cup I).$$

According to Lerman [25] one calls an element a of a lattice $\langle L, \vee, \wedge, 0, 1\rangle$ L-simple if for all $b \in L$

$$a \wedge b = 0 \Rightarrow b = 0.$$

For the considered lattice $\mathfrak{E}^*(\beta)$ this definition says that the equivalence class of a β-r.e. set W is $\mathfrak{E}^*(\beta)$-simple iff every β-r.e. set $U \subseteq L_\beta - W$ is l-finite.

It is customary in recursion theory to exclude from this lattice theoretic definition the largest element 1, i.e. the class of β-r.e. sets with l-finite complement. Thus we arrive at the following definition, which coincides for admissible ordinals with the standard definition in α-recursion theory (see Lerman [25]).

Definition 9.3. A β-r.e. set W is simple if $L_\beta - W$ is not l-finite but every β-r.e. subset of $L_\beta - W$ is l-finite.

If β is admissible or weakly inadmissible a β-r.e. set W is simple iff $L_\beta - W$ is not i-finite but every β-r.e. subset of $L_\beta - W$ is i-finite (because an i-finite set which is not l-finite contains a β-r.e. non β-recursive set).

We consider in Section 10 simple sets for strongly inadmissible β with β^* β-recursively regular. For these β a β-r.e. set W with $L_\beta - W \subseteq \beta^*$ is simple iff $L_\beta - W$ is unbounded in β^* but every β-r.e. set $U \subseteq L_\beta - W$ is bounded below β^* and this holds iff $L_\beta - W$ is unbounded in β^* but every $U \in L_\beta$ with $U \subseteq L_\beta - W$ is bounded below β^*.

10. A β-r.e. degree without a simple set and a splitting theorem for simple β-r.e. sets

If β is admissible or weakly inadmissible then every β-r.e. non β-recursive i-degree contains a simple set. This comes as a side result out of the regular set theorem in α-recursion theory, which says that every α-r.e. α-degree contains an α-r.e. set A which is regular (i.e. $\forall \delta < \alpha\ (A \cap L_\delta \in L_\alpha)$) (see [28]). Now if $f: \alpha \to A$ is an α-recursive enumeration of a regular non α-recursive set A, then the deficiency set

$$D := \{x \mid \exists\, y > x\ (f(y) < f(x))\}$$

is simple and of the same α-degree as A.

For strongly inadmissible β regular β-r.e. sets are very rare. E.g. for $\beta = \aleph_1 + \omega$ every regular β-r.e. set is of degree $\mathbf{0}_i$. But for these β one can still construct nontrivial simple sets. In fact a solution of Post's problem in general produces automatically simple sets. Theorem 10.4 below shows that these simple sets have some of the benefits of regular sets in α-recursion theory: we can split them into two r.e. sets of lower degree. On the other hand the question which β-r.e. i-degrees contain simple sets is more difficult than the analogous question about regular sets in α-recursion theory. We produce in Theorem 10.3 a β-r.e. non β-recursive i-degree without a simple set. Both constructions rely on the following combinatorial lemma.

Lemma 10.1 (ZFC). *Assume ρ and κ are cardinals and $\rho < \operatorname{cf} \kappa$. Let M be an unbounded subset of κ and f be a function which assigns to every element of M some subset of κ of cardinality less than ρ. Then there is some $\delta < \kappa$ such that*

$$\forall \sigma < \kappa\, \exists \tau \geqslant \sigma\, (\tau \in M \wedge f(\tau) \cap (\sigma - \delta) = \emptyset).$$

Proof. Assume such a δ does not exist. Then

$$\forall \delta < \kappa\, \exists \sigma < \kappa\, \forall \tau \geqslant \sigma\, (\tau \in M \rightarrow f(\tau) \cap (\sigma - \delta) \neq \emptyset).$$

Define a strictly increasing sequence $(\delta_\gamma)_{\gamma < \rho}$ of ordinals less than κ as follows:

$\delta_0 := 0;$

$\delta_{\gamma+1}$ is the least $\sigma > \delta_\gamma$ such that $\sigma < \kappa$ and

$\forall \tau \geqslant \sigma\, (\tau \in M \rightarrow f(\tau) \cap (\sigma - \delta_\gamma) \neq \emptyset);$

$\delta_\lambda := \sup_{\gamma < \lambda} \delta_\gamma$ for λ limit.

Since $\rho < \operatorname{cf} \kappa$ this sequence is well defined and $\tilde{\delta} := \sup_{\gamma < \rho} \delta_\gamma$ is less than κ. Take some $\tilde{\tau} \in M$ such that $\tilde{\tau} \geqslant \tilde{\delta}$. Then

$$\forall \gamma < \rho\, (f(\tilde{\tau}) \cap (\delta_{\gamma+1} - \delta_\gamma) \neq \emptyset).$$

But this is impossible since the cardinality of $f(\tilde{\tau})$ is less than ρ. □

The preceding lemma is closely related to the familiar Δ-System lemma, which is often used in forcing and combinatorics (see Kunen [23] and Jech [16]). In fact Lemma 10.1 supplies a different proof of the Δ-System lemma, which follows as a corollary.

Corollary 10.2. *Assume v is a cardinal with $2^v = v^+$. Then for every family W of subsets of v^{++} of cardinality $\leqslant v$ with $|W| = v^{++}$ there is some $W_1 \subseteq W$ such that $|W_1| = |W|$ and W_1 is quasi-disjoint (i.e. there is some z such that $\forall x, y \in W_1\, (x \cap y = z)$.*

Proof. Let f be an enumeration of W with $\operatorname{dom} f = v^{++}$. By Lemma 10.1 there is some $\delta < v^{++}$ such that

$$\forall \delta < \kappa \exists \tau \geqslant \sigma\, (f(\tau) \cap (\sigma - \delta) = \emptyset).$$

Thus we can choose some $W' \subseteq W$ such that $|W'| = |W|$ and for all $x, y \in W'$ $x \cap y \subseteq \delta$. Since $|\delta| \leqslant v^+$ and $(v^+)^v = v^+ < v^{++}$ there is some $W_1 \subseteq W'$ such that $|W_1| = |W'|$ and all elements of W_1 have the same intersection with δ. □

Observe that Lemma 10.1 works as well if κ is the successor or a singular cardinal ρ. This situation will actually occur in the application of Lemma 10.1 in Theorem 10.3 and Theorem 10.4 (e.g. for $\beta = \aleph_{\omega+1} + \aleph_1$). One can not use the Δ-System lemma here, because it does not hold for $\kappa = \rho^+$, ρ singular, W a collection of sets of size less than ρ, card $W = \kappa$.

Theorem 10.3. *Assume $\sigma 1 \operatorname{cf} \beta < \beta^*$ and β^* is a regular cardinal in L. Then there is a β-r.e. i-degree $\boldsymbol{a} > \boldsymbol{0}$ such that no i-degree $\boldsymbol{b} \leq \boldsymbol{a}$ contains a simple set.*

Proof. We use the combinatorial Lemma 10.1 with $\kappa := \beta^*$ and $\rho := \sigma 1 \operatorname{cf} \beta$ and in addition $\diamondsuit$. We construct a β-r.e. non β-recursive set $A \subseteq \beta^*$ such that $B \not\leq_i A$ for all simple sets B. We can assume without loss of generality that $L_\beta - B \subseteq \beta^*$. If D is any other simple set we consider instead $B := P[D] \cup (L_\beta - \beta^*)$, where P maps L_β one-one onto β^*. B is again simple and of the same i-degree as D.

If B is a simple set with $L_\beta - B \subseteq \beta^*$, then $L_\beta - B$ is unbounded in β^* but every β-r.e. set $U \subseteq L_\beta - B$ is bounded below β^*. If $B = \{e\}^A$, then there is a stage of the construction where $\{e\}$ computes from the so far enumerated part of A, that unboundedly many $x \in \beta^*$ are not in B (we use here that $\sigma 1 \operatorname{cf} \beta < \beta^*$). The strategy is then, to preserve these computations for unboundedly many arguments x. The set of these arguments is then an element of L_β and unbounded in $L_\beta - B$. This contradicts B simple.

The burden of this strategy is, that a single requirement may prevent an unbounded subset of β^* from A. Thus we get problems to satisfy the positive requirements which make A non β-recursive.

Our escape is the fact that not all unbounded subsets of β^* are equal. If we consider e.g. complements of closed unbounded sets in β^*, then these complements are so thin, that the union of less than β^* many still does not fill up β^*. By the help of Lemma 10.1 we may choose an unbounded set of arguments x such that all the computations $\{e\}^A(x)$ together use only some part of A which lies in the complement of a closed unbounded set. Concerning the $\delta < \beta^*$ given by Lemma 10.1 we use $\diamondsuit$ in order to guess at $A \cap \delta$.

Fix a $\diamondsuit$-sequence $\langle S_\delta \mid \delta \in \beta^* \rangle \in L_\beta$ as in the proof of Theorem 7.1. Let P be a β-recursive function which maps L_β one-one onto β^*. Let $q : \sigma 1 \operatorname{cf} \beta \to \beta$ be β-recursive, strictly increasing and cofinal.

At stage $\delta < \beta^*$ we consider requirements $N_{e,X}$ and P_e where $e < \delta$ and $X \in S_\delta$. The cardinality of these requirements is less than β^*. $N_{e,X}$ tries to prevent that $B \leq_i A$ via $P^{-1}(e)$ for a simple set B. $N_{e,X}$ uses X as a guess at $A \cap \delta$. P_e tries to prevent that $W_{P^{-1}(e)} = L_\beta - A$.

The construction proceeds in $\sigma 1 \operatorname{cf} \beta$ many steps. At step γ we use all information available in $L_{q(\gamma)}$. At every step γ we run through all stages $\delta < \beta^*$ and consider all requirements at stage δ in some fixed order.

Step γ, stage δ: Assume $N_{e,X}$ is the next requirement which is to be considered. We do only something for $N_{e,X}$, if $P^{-1}(e)$ converges in $L_{q(\gamma)}$ and if never before something was done for $N_{e,X}$ at stage δ. In this case we only do something if ⊛ holds:

$$⊛ \qquad \forall\, \sigma < \beta^* \,\exists\, \tau \geq \sigma\, (\tau \in \beta^* \wedge \exists \text{ i-finite } H\, (\langle \tau, H\rangle \in W^{L_{q(\gamma)}}_{P^{-1}(e)} \wedge H \cap (\sigma - \delta) = \emptyset \wedge H \cap X = \emptyset \wedge (\text{no element of } H \text{ is already in } A)).$$

If ⊛ holds we choose a sequence $\langle \tau_\nu, H_\nu\rangle_{\nu<\beta^*}$ such that $\langle \tau_\nu\rangle_{\nu<\beta^*}$ and $\langle \min(H_\nu - \delta)\rangle_{\nu<\beta^*}$ are strictly increasing and for every $\nu\, H_\nu$ is i-finite, $\langle \tau_\nu, H_\nu\rangle \in W^{L_{q(\gamma)}}_{P^{-1}(e)}$, $H_\nu \cap X = \emptyset$, no element of H_ν is already in A and such that $\bigcup\{H_\nu \mid \nu < \beta^*\}$ lies in the complement of a closed unbounded subset of β^*. One gets such a sequence by constructing simultaneously a closed unbounded set in the complement of $\bigcup\{H_\nu \mid \nu<\beta^*\}$, using ⊛. After we have fixed such a sequence we restrain $(\bigcup\{H_\nu \mid \nu < \beta^*\}) - \delta$ from A for $N_{e,X}$ at stage δ.

If P_e is the next requirement at stage δ, we only do something if no element of $W^{L_{q(\gamma)}}_{P^{-1}(e)}$ has so far been enumerated in A. If there is some $x \in W^{L_{q(\gamma)}}_{P^{-1}(e)}$, $\delta < x < \beta^*$, which is not restrained from A for any requirement at some stage $\leq \delta$, we enumerate the least such x in A.

End of the construction.

In order to see that A has the desired properties we note first that for every stage δ the set of x which we enumerated in A for requirements at stage $\leq \delta$ is bounded below β^*. Since δ is not enumerated in A for a requirement at stage δ and $\omega < \operatorname{cf} \beta^*$ there is a closed unbounded set C in β^* which is disjoint from A. Further for every stage δ there is a closed unbounded set C_δ in β^* such that no element of C_δ is ever restrained from A for a requirement at stage $\leq\delta$ (use the fact that the intersection of $<\beta^*$ many club sets in β^* is club).

Assume for a contradiction that $L_\beta - A = W_{P^{-1}(e)}$. Consider some stage $\delta > e$. Then $C_\delta \cap C$ is closed unbounded in β^* and $C_\delta \cap C \subseteq W_{P^{-1}(e)}$. Take any $x \in C_\delta \cap C$ with $x > \delta$. Take γ large enough such that $P^{-1}(e)$ converges in $L_{q(\gamma)}$ and $x \in W^{L_{q(\gamma)}}_{P^{-1}(e)}$. Then we make $W_{P^{-1}(e)} \cap A \neq \emptyset$ during the consideration of P_e at stage δ of step γ if this was not already done before. Therefore A is not β-recursive.

Assume now for a contradiction that $B \leq_{\mathrm{i}} A$ via $P^{-1}(e)$ for some simple set B with $L_\beta - B \subseteq \beta^*$. Then there is some stage γ_0 such that $P^{-1}(e)$ converges in $L_{q(\gamma_0)}$ and there is an unbounded set of $\tau \in \beta^* - B$ such that for some i-finite H with $H \subseteq L_\beta - A$ we have $\langle \tau, H\rangle \in W^{L_{q(\gamma)}}_{P^{-1}(e)}$ (if such a γ_0 does not exist we get a cofinal function from $\sigma 1 \operatorname{cf} \beta$ in β^*, contradicting the regularity of β^*). We apply then Lemma 10.1 to the function which assigns to these τ some H as above. Thus there is some $\delta < \beta^*$ such that for all $\sigma < \beta^*$ we can find some τ where the associated H satisfies $H \cap (\sigma - \delta) = \emptyset$. Go to some stage $\delta_0 \geq \delta$ such that $A \cap \delta_0 \in S_{\delta_0}$. At stage γ_0 we consider then $N_{e,A\cap\delta_0}$ at stage δ_0. For this requirement ⊛ is at this point satisfied. Therefore at some point of the construction a sequence $\langle \tau_\nu, H_\nu\rangle_{\nu\in\beta^*}$ is associated with $N_{e,A\cap\delta_0}$ at stage δ_0. Only boundedly much is enumerated in A for positive requirements at stages $\leq\delta_0$. Therefore there is some

$\nu_0 < \beta^*$ such that $H_\nu \subseteq L_\beta - A$ for $\nu \geqslant \nu_0$. Since $B \leqslant_i A$ via $P^{-1}(e)$ we have then $\tau_\nu \notin B$ for $\nu \geqslant \nu_0$. Thus $\{\tau_\nu \mid \nu \geqslant \nu_0\}$ is an unbounded subset of $L_\beta - B$ which is an element of L_β. This contradicts B simple.

This finishes the proof of Theorem 10.3.

Theorem 10.4. *Assume $\sigma 1 \operatorname{cf} \beta < \beta^*$ and β^* is Σ_2-regular (i.e. there is no $\Sigma_2 L_\beta$ function which maps some $\delta < \beta^*$ cofinally into β^*). Then for every simple β-r.e. set W there are β-r.e. sets W_1, W_2 such that $W_1 \cap W_2 = \emptyset$, $W = W_1 \cup W_2$, $W_1 <_i W$, $W_2 <_i W$ and $W =_i W_1 \oplus W_2$.*

Proof. No Splitting theorem has so far been proved for the strongly inadmissible case because the familiar preservation strategy of Sacks leads to many problems. Fix some β-recursive strictly increasing function q which maps $\sigma 1 \operatorname{cf} \beta$ cofinally into β. It may very well be that for every $\gamma < \sigma 1 \operatorname{cf} \beta$ the γth requirement demands that from a certain point on an initial segment of W_1 of length $q(\gamma)$ has to be protected. Since every element of W must end up in W_1 or in W_2 this means that nearly no computation from W_2 can be protected for the sake of a requirement of priority $\geqslant \sigma 1 \operatorname{cf} \beta$. If β is admissible or weakly inadmissible one can make the list of requirements so short, that no requirement of priority $\geqslant \sigma 1 \operatorname{cf} \beta$ exists (see [15, 31, 40, 41]). There is no way to do this here because β may have a larger cardinality than $\sigma 1 \operatorname{cf} \beta$.

Our first step is to project the problem into β^* so that we can make use of the assumed regularity of β^*. It is obviously enough to solve the following problem:

Given some β-r.e. $C \subseteq \beta^*$ such that $\beta^* - C$ is unbounded in β^* but every β-r.e. set $U \subseteq \beta^* - C$ is bounded below β^*. Construct β-r.e. sets A, B such that $A \cap B = \emptyset$, $C = A \cup B$, $A <_i C$, $B <_i C$ and $C =_i A \oplus B$.

As in the proof of Theorem 10.3, strong inadmissibility of β has one good feature: If $C = \{e\}^A$, then there is a stage in the construction where for an in β^* unbounded set of arguments there exist computations of $\{e\}$ from the constructed part of A. At this point we have then a lot of choice concerning which computations from A we want to preserve.

But the similarity to the proof of Theorem 10.3 ends here because we can no longer afford to restrain forever any in β^* unbounded set of elements from A for a single requirement, no matter how clever we choose this unbounded set. Every element of C is put in A or in B and all the elements which are restrained from A will injure computations from B.

Fortunately in the considered situation the Sacks preservation strategy does not require to preserve *forever* a *large* number of computations for a single requirement (although this has so far always be done). For a single requirement it is enough to preserve for a limited time a large number of equations $C(x) = \{e\}^A(x)$. This preservation finally forces the appearance of an inequality for some x_0 which is enumerated in C. From that point on we only have to preserve the single computation $\{e\}^A(x_0)$, which uses only i-finitely much from A. The elements which were restrained in order to preserve the other computations $\{e\}^A(x)$ are released and can now be restrained from A for the sake of other requirements. A combinatorial trick makes sure that no single element is restrained unboundedly often for changing requirements. Therefore we can finally put every element of C in A or B.

In order to make the combinatorial argument work it is essential that we choose very carefully the large number of equations which we preserve for a limited time. Lemma 10.1 supplies again a stage $\delta<\beta^*$ where a particularly convenient choice is possible.

Observe that it is not possible to make A, B in addition simple because each lies in the complement of the other.

We start now with exact construction of A, B and assume that a $\Sigma_1 L_\beta$ definition of C has been fixed.

Let P be a β-recursive function which maps L_β one-one onto β^*. Fix a $\diamond$-sequence $\langle S_\delta \mid \delta<\beta^*\rangle \in L_\beta$ as in the proof of Theorem 7.1.

We have for every $e\in\beta^*$ requirements N_e^A, N_e^B. N_e^A tries to prevent that

$$\forall x\in\beta^*\,(x\notin C \leftrightarrow \exists\text{ i-finite } H\,(\langle x, W\rangle \in W_{P^{-1}(e)} \wedge H\subseteq \beta^*-A)).$$

On stage $\delta<\beta^*$ we consider all requirements $N_{e,X}^A$, $N_{e,X}^B$ with $e<\delta$ and $X\in S_\delta$. $N_{e,X}^A$ considers X as a guess at $A\cap\delta$.

There is a β-recursive function h which maps for every δ the requirements considered at stage δ one-one onto some ordinal $h(\delta)$ with $\delta\leq h(\delta)<\beta^*$.

We write then the requirements at stage δ in the form $(R_{\delta,j})_{j<h(\delta)}$.

The construction proceeds like in Theorem 10.3 in $\sigma 1\operatorname{cf}\beta$ many steps. At every step $\gamma<\sigma 1\operatorname{cf}\beta$ we run through all stages δ and consider the requirements $(R_{\delta,j})_{j<h(\delta)}$ in their assigned order.

Construction:

Step γ, *Stage* δ. Assume $R_{\delta,j}$ is the next requirement to be considered.

Assume that $R_{\delta,j}$ is some $N_{e,X}^A$. We do only something for $N_{e,X}^A$ if $L_{q(\gamma)}\vDash [\exists y(P(y)=e)]$.

If $R_{\delta,j}$ was never activated before we activate $R_{\delta,j}$ now if

$$(*)\qquad \forall\sigma(\delta<\sigma<\beta^*\to\exists\tau(\sigma\leq\tau<\beta^*\wedge\exists\text{ i-finite } H(\langle\tau,H\rangle\in W_{P^{-1}(e)}^{L_{q(\gamma)}}$$
$$\wedge(\text{no element of } H \text{ is already in } A)$$
$$\wedge H\subseteq(\beta^*-\sigma)\cup\delta\wedge H\cap X=\emptyset))).$$

In this case we associate a sequence $\langle\tau_\nu, H_\nu\rangle_{\nu<\beta^*}$ with $R_{\delta,j}$ which is defined by recursion as follows. Assume $\langle\tau_{\nu'}, H_{\nu'}\rangle_{\nu'<\nu}$ is already defined and $\nu<\beta^*$. Define

$$\sigma_\nu:=\sup\{y\in\beta^* \mid (\text{some sequence } \langle\tilde\tau_\rho,\tilde H_\rho\rangle_{\rho<\beta^*} \text{ has been associated with a requirement on stage } {<}\delta \text{ or some } R_{\delta,j'}, \text{ with } j'<j \text{ and } y\in\tilde H_\rho \text{ for some } \rho\leq\nu)\vee(y=\tau_{\nu'} \text{ for some } \nu'<\nu)\vee(y\in H_{\nu'} \text{ for some } \nu'<\nu)\}.$$

We define then $\langle\tau_\nu, H_\nu\rangle$ as a pair $\langle\tau, H\rangle$ which has a relationship to $\sigma:=\max(\sigma_\nu,\delta)+1$ as in $(*)$.

In case that $R_{\delta,j} \equiv N^A_{e,X}$ was already activated at some step before but is not yet finished we check first whether some element of $\delta - X$ has been enumerated into A. If yes, we finish $R_{\delta,j}$ now (nothing else is done for $R_{\delta,j}$). If no, we check for the sequence $\langle \tau_\nu, H_\nu \rangle$ which was associated with $R_{\delta,j}$ before whether for some $\nu < \beta^*$ $L_{q(\gamma)} \models [\tau_\nu \in C]$ and no element of $H_\nu - \delta$ is already in A or has before been restrained from A or B for some requirement on stage $\leq \delta$. If such a ν exists we choose it minimal and restrain $H_\nu - \delta$ from A for $R_{\delta,j}$. $R_{\delta,j}$ is then finished.

At the end of step γ we consider all y such that $L_{q(\gamma)} \models [y \in C]$ but y is not yet enumerated in A or B. If y is not in some $H_\nu - \delta$ for some sequence $\langle \tau_\nu, H_\nu \rangle_{\nu < \beta^*}$ associated with some activated but not yet finished requirement $R_{\delta,j}$ we enumerate now y in A or in B. We look then at the requirement of highest priority for which y has been restrained. If this requirement restrains y from A then we enumerate y in B. If it restrains y from B or if y is not restrained for any requirement we enumerate y in A.

We say that $R_{\delta,j}$ has higher priority than $R_{\delta',j'}$ if $\delta < \delta'$ or $\delta = \delta'$ and $j < j'$.

End of the construction.

Fact 1. *Every requirement $R_{\delta,j} \equiv N^A_{e,X}$ which is activated is later finished.*

Proof. Let $\langle \tau_\nu, H_\nu \rangle_{\nu < \beta^*}$ be the sequence associated with $R_{\delta,j} \equiv N^A_{e,X}$ when it is activated.

Obviously $R_{\delta,j}$ is finished if some element of $\delta - X$ is enumerated in A. Thus we can assume that $A \cap \delta \subseteq X$. It is already enough to know that β^* is Σ_1-regular in order to see that there is some $\nu_0 < \beta^*$ such that for $\nu \geq \nu_0$ no element of $H_\nu - \delta$ is ever restrained for some requirement on stage $\leq \delta$ (of course we use here as well that $H_\nu - \delta$ contains for large ν only large elements). Further as long as $R_{\delta,j}$ is not finished no element of any $H_\nu - \delta$ is enumerated into A according to the construction.

Therefore as soon as γ is large enough such that $L_{q(\gamma)} \models [\tau_\nu \in C]$ for some $\nu \geq \nu_0$ (such a γ exists by the properties of C) $R_{\delta,j}$ will be finished. □

Fact 2. *For every element of $y < \beta^*$ there are only finitely many requirements $R_{\delta,j}$ such that $y \in H_\nu - \delta$ for some $\nu < \beta^*$, where $\langle \tau_\nu, H_\nu \rangle_{\nu < \beta^*}$ is the sequence associated with $R_{\delta,j}$.*

Proof. If y is in some $\tilde{H}_\rho - \tilde{\delta}$ of a sequence $\langle \tilde{\tau}_\nu, \tilde{H}_\nu \rangle$ associated with $R_{\tilde{\delta},\tilde{j}}$ and y is as well in some $H_\nu - \delta$ of a later defined sequence $\langle \tau_\nu, H_\nu \rangle$ associated with a requirement $R_{\delta,j}$ of lower priority than $R_{\tilde{\delta},\tilde{j}}$, then $\nu < \rho$ by construction (see the definition of σ_ν in the construction). Therefore this cannot be iterated infinitely often.

But still there remains something to prove because we may have $\nu \geq \rho$ if $R_{\delta,j}$ is of higher priority than $R_{\tilde{\delta},\tilde{j}}$.

Assume the claim is false for y. Let M be the set of the first ω requirements $R_{\delta,j}$ (ordered by priority) such that $y \in H_\nu - \delta$ for some $\nu < \beta^*$, where $\langle \tau_\nu, H_\nu \rangle_{\nu < \beta^*}$ is the sequence associated with $R_{\delta,j}$. We consider then out of M the first ω requirements $(R_{\delta_i,j_i})_{i\in\omega}$ in the order of their activation during the construction. This sequence contains a subsequence $(R_{\delta_{i_n},j_{i_n}})_{n\in\omega}$ such that for every $n\in\omega\, R_{\delta_{i_n},j_{i_n}}$ has higher priority than $R_{\delta_{i_{n+2}},j_{i_{n+2}}}$ (we use here the definition of M). If $\langle\tau_\nu; H_\nu\rangle_{\nu<\beta^*}$ is the sequence associated with $R_{\delta_{i_n},j_{i_n}}$ we write than ν_{i_n} for the ν with $y\in H_\nu-\delta_{i_n}$. Then by construction we have $\nu_{i_0}>\nu_{i_1}>\cdots$, a contradiction. □

We show now that $C=\{P^{-1}(e)\}^A$ implies that the requirement N_e^A succeeds at some stage $\delta>e$, which leads to a contradiction. In order to prove the existence of such a stage δ one can directly apply the combinatorial Lemma 10.1 if β^* is sufficiently regular. In order to get along only with Σ_2-regularity of β^* we give in the proof of Fact 4 below an effective version of the proof of the combinatorial lemma. The following Fact 3 will be needed for this effective version.

Fact 3. *Assume that for all* $x\in\beta^*$

$$x\in\beta^*-C\Leftrightarrow\exists\text{ i-finite } H\,(\langle x,H\rangle\in W_{P^{-1}(e)}\wedge H\subseteq\beta^*-A).$$

Then for every $\gamma_0<\sigma 1\operatorname{cf}\beta$ *there is some* γ *such that*

$$\gamma_0\leq\gamma<\sigma 1\operatorname{cf}\beta$$

and

$$\forall\delta<\beta^*\,\forall\sigma<\beta^*\,\exists\tau\,(\sigma\leq\tau<\beta^*\wedge\exists\text{ i-finite } H\,(\langle\tau,H\rangle\in W_{P^{-1}(e)}^{L_{\alpha(\gamma)}}\wedge H\cap A\cap\delta=\emptyset\wedge\text{(no element of } H \text{ is enumerated in } A \text{ by the end of step } \gamma))).$$

Proof. Assume the contrary. Then for every γ with $\gamma_0\leq\gamma<\sigma 1\operatorname{cf}\beta$ there are $\delta_\gamma,\sigma_\gamma<\beta^*$ such that no $\tau\geq\sigma_\gamma$ satisfies the condition above. It is enough to show that the σ_γ can be chosen in such a way that $\tilde\sigma:=\sup\{\sigma_\gamma\mid\gamma_0\leq\gamma<\sigma 1\operatorname{cf}\beta\}<\beta^*$. Because then we can take some $\tau>\tilde\sigma$ with $\tau\in\beta^*-C$. For this τ there exists some i-finite H with $\langle\tau,H\rangle\in W_{P^{-1}(e)}^{L_{\alpha(\gamma)}}$ for some $\gamma_1\geq\gamma_0$ and $H\subseteq\beta^*-A$. Because $\tau\geq\sigma_{\gamma_1}$ this is a contradiction to the properties of σ_{γ_1}.

We show now that one can assign ordinals δ_γ, σ_γ as above by a Σ_2 function. In order to express that δ_γ, σ_γ have the correct properties we need as well $\delta_\gamma\cap A$. In order to see that the function $\delta\to\delta\cap A$ is $\Sigma_2 L_\beta$ we consider the β-recursive set

$$A_r:=\{\langle v,x\rangle\mid v<\sigma 1\operatorname{cf}\beta\text{ and } x \text{ is enumerated in } A \text{ by the end of step } v\}.$$

For every $\delta<\beta^*$ the set $A_r\cap(\sigma 1\operatorname{cf}\beta\times\delta)$ is an element of L_β and the function $\delta\to A_r\cap(\sigma 1\operatorname{cf}\beta\times\delta)$ is $\Sigma_2 L_\beta$. One can express therefore by a Σ_2 formula that for $\gamma<\sigma 1\operatorname{cf}\beta$ some tuple $\langle x_1,x_2,x_3,x_4\rangle$ has the properties which we expect from $\langle\delta_\gamma,\sigma_\gamma,A_r\cap(\sigma 1\operatorname{cf}\beta\times\delta_\gamma),A\cap\delta_\gamma\rangle$. With Σ_2-uniformization we get a $\Sigma_2 L_\beta$ function $\gamma\to\langle\delta_\gamma,\sigma_\gamma,A_r\cap(\sigma 1\operatorname{cf}\beta\times\delta_\gamma),A\cap\delta_\gamma\rangle$ and thus in particular a $\Sigma_2 L_\beta$ function $\gamma\to\sigma_\gamma$. Since β^* is Σ_2-regular the set $\{\sigma_\gamma\mid\gamma_0\leq\gamma<\sigma 1\operatorname{cf}\beta\}$ is bounded below β^*. □

In the following let p be an element of L_β which contains all parameters of the construction. We know then that $A\cap\delta$, $B\cap\delta\in S_\delta$ for $\delta\in C_p:=\{\delta<\beta^* \mid h_1[(\delta\cup\{p\})\times\omega]\cap\beta^*=\delta\}$ as in the proof of Theorem 7.1.

Fact 4. *Assume that for all* $x\in\beta^*$

$$x\in\beta^*-C\Leftrightarrow\exists\ i\text{-finite}\ H\ (\langle x,H\rangle\in W_{p^{-1}(e)}\wedge H\subseteq\beta^*-A).$$

Then for every $\gamma_0<\sigma 1\operatorname{cf}\beta$ *and every* $\delta_0<\beta^*$ *there are* γ,δ *such that* $\gamma_0\leq\gamma<\sigma 1\operatorname{cf}\beta$, $\delta_0\leq\delta<\beta^*$, $\delta\in C_p$ *and*

$$\forall\sigma<\beta^*\exists\tau\ (\sigma\leq\tau<\beta^*\wedge\exists\ \text{i-finite}\ H\ (\langle\tau,H\rangle\in W_{p^{-1}(e)}^{L_{q(\gamma)}}$$
$$\wedge H\cap A\cap\delta=\emptyset\wedge(\text{no element of } H \text{ is enumerated}$$
$$\text{in } A \text{ by the end of stage } \gamma)\wedge H\cap(\sigma-\delta)=\emptyset)).$$

Proof. Take $\gamma\geq\gamma_0$ according to Fact 3. Assume for a contradiction that the desired δ does not exist for this γ. Then we can assign to every $\delta\in C_p$ some $\sigma\in C_p$ such that $\delta<\sigma<\beta^*$ and for all $\langle\tau,H\rangle\in W_{p^{-1}(e)}^{L_{q(\gamma)}}$ with $\sigma\leq\tau<\beta^*$, $H\cap A\cap\delta=\emptyset$ and no element of H is enumerated in A by the end of stage γ we have $H\cap(\sigma-\delta)\neq\emptyset$. We define a function $h:\sigma 1\operatorname{cf}\beta\to\beta^*$ such that for $\nu<\sigma 1\operatorname{cf}\beta$ $h(\nu+1)$ has the same relationship to $h(\nu)$ as σ to δ above. We define $h(\lambda)=\sup_{\nu<\lambda}h(\nu)$ for limit ordinals λ. In order to show that $h\in L_\beta$ we first define a function g which assigns to ν not only an ordinal $\sigma=h(\nu)$ but in addition

$$\langle A_r\cap(\sigma 1\operatorname{cf}\beta\times\sigma),\ A\cap\sigma,\ (\beta^*-C_p)_r\cap(\sigma 1\operatorname{cf}\beta\times\sigma),\ (\beta^*-C_p)\cap\sigma\rangle.$$

A_r is again a β-recursive set associated with A (see the proof of Fact 3) and $(\beta^*-C_p)_r$ is an analogous β-recursive set associated with the β-r.e. set β^*-C_p. The exact definition of g is as follows. Fix some $\delta_1\geq\delta_0$ such that $\delta_1\in C_p$.

$$g(0):=\langle\delta_1,A_r\cap(\sigma 1\operatorname{cf}\beta\times\delta_1),A\cap\delta_1,(\beta^*-C_p)_r$$
$$\cap(\sigma 1\operatorname{cf}\beta\times\delta_1),(\beta^*-C_p)\cap\delta_1\rangle;$$

$$g(\nu+1):=\langle\sigma,A_r\cap(\sigma 1\operatorname{cf}\beta\times\sigma),A\cap\sigma,(\beta^*-C_p)_r$$
$$\cap(\sigma 1\operatorname{cf}\beta\times\sigma),(\beta^*-C_p)\cap\sigma\rangle$$

where $\sigma>(g(\nu))_0$ has the same properties w.r.t. $(g(\nu))_0$ as σ w.r.t. δ above and is minimal with this property;

$$g(\lambda):=\langle\sigma,A_r\cap(\sigma 1\operatorname{cf}\beta\times\sigma),A\cap\sigma,(\beta^*-C_p)_r$$
$$\cap(\sigma 1\operatorname{cf}\beta\times\sigma),(\beta^*-C_p)\cap\sigma\rangle$$

where $\sigma=\sup_{\nu<\lambda}(g(\nu))_0$ for limit ordinals λ.

In order to show that $g\in L_\beta$ we prove by induction on ν ($\nu\leq\sigma 1\operatorname{cf}\beta$) that $g\restriction\nu\in L_{\beta^*}$. The induction step is only nontrivial if ν is a limit ordinal λ. In this case one knows already that $g\restriction\rho\in L_{\beta^*}$ for $\rho<\lambda$ and one can easily see that the function $\rho\to g\restriction\rho$ for $\rho<\lambda$ is Σ_2 definable over L_β. One has to express in this Σ_2

formula that for $\nu+1<\rho$ $(g(\nu+1))_0$ is minimal such that $(g(\nu+1))_0\in C_p$ and.... At this point we use that the corresponding initial segments of C_p are included in the values of g. So $g\restriction\lambda$ is $\Sigma_2 L_\beta$ and therefore $\varepsilon:=\sup\{(g(\nu))_0 \mid \nu<\lambda\}<\beta^*$ because β^* is Σ_2 regular. But then we can give as well a $\Sigma_1 L_\beta$ definition of $g\restriction\lambda$ because we can use the sets $A\cap\epsilon$, $C_p\cap\epsilon\in L_\beta$ as parameters in this $\Sigma_1 L_\beta$ definition. Since $g\restriction\lambda$ is $\Sigma_1 L_\beta$ and bounded below β^* it is in fact an element of L_{β^*}.

Thus we have shown that $g\in L_\beta$. Therefore $\tilde{\delta}:=\sup\{(g(\nu))_0 \mid \nu<\sigma 1 \operatorname{cf}\beta\}<\beta^*$. Since γ was chosen according to Fact 3 there is a pair $\langle\tau, H\rangle\in W^{L_{q(\gamma)}}_{P^{-1}(e)}$ such that $\tilde{\delta}<\tau<\beta^*$, H i-finite $H\cap A\cap\tilde{\delta}=\emptyset$ and no element of H is enumerated in A by the end of step γ. By the definition of g we have for every $\nu<\sigma 1 \operatorname{cf}\beta$ $H\cap((g(\nu+1))_0-(g(\nu))_0)\neq\emptyset$. Since the function $\nu\rightarrow(g(\nu))_0$ is strictly increasing this is a contradiction to H being i-finite. □

Fact 5. *There is no $e<\beta^*$ such that for all $x\in\beta^*$*

$$x\in\beta^*-C\Leftrightarrow\exists\ \text{i-finite}\ H\ (\langle x, H\rangle\in W_{P^{-1}(e)}\wedge H\subseteq\beta^*-A).$$

Proof. Assume the contrary. Take some $\gamma_0<\sigma 1 \operatorname{cf}\beta$ such that $L_{q(\gamma_0)}\models \exists y\,(P(y)=e)$ and some δ_0 such that $e<\delta_0<\beta^*$. For these γ_0, δ_0 let γ, δ be ordinals as in Fact 4.

Since $\delta\in C_p$ we have $A\cap\delta\in S_\delta$. Further $N^A_{e,A\cap\delta}$ on stage δ is activated at step γ if it was not already activated before. According to Fact 1 there is then some step γ_1 where $N^A_{e,A\cap\delta}$ on stage δ is finished. Since no element of $\delta-(A\cap\delta)$ is enumerated into A the requirement is finished because there is a pair $\langle\tau_\nu, H_\nu\rangle$ out of the sequence associated with $N^A_{e,A\cap\delta}$ on stage δ such that $L_{q(\gamma_1)}\models[\tau_\nu\in C]$ and no element of $H_\nu-\delta$ is restrained for some requirement on stage $\leqslant\delta$ and no element of $H_\nu-\delta$ is already in A. all elements of $H_\nu-\delta$ are then restrained from A for $N^A_{e,A\cap\delta}$ on stage δ when this requirement is finished.

Since $\delta\in C_p$ no element of $H_\nu-\delta$ is ever restrained for a requirement on stage $<\delta$. Further no element of $H_\nu-\delta$ is ever restrained for a requirement on stage $\geqslant\delta$. after it is restrained for $N^A_{e,A\cap\delta}$ on stage δ. Therefore no element of $H_\nu-\delta$ is ever restrained for a requirement of higher priority.

Thus $H_\nu\subseteq\beta^*-A$ and $\tau_\nu\in C$ and $\langle\tau_\nu, H_\nu\rangle\in W_{P^{-1}(e)}$, a contradiction. □

Fact 6. $C=A\cup B$.

Proof. By Fact 1 and Fact 2 for every $y\in\beta^*$ there is a step after which y is never restrained by an activated requirement which is not yet finished. □

It follows as in classical recursion theory that $A\leqslant_i C$, $B\leqslant_i C$ and $C=_i A\oplus B$. Therefore by Fact 5 and Fact 6 A and B have all the desired properties. This finishes the proof of Theorem 10.4. □

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