PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, June 1981

A COUNTABLE BASIS FOR Σ_2^1 SETS AND RECURSION THEORY ON $\aleph_1$

WOLFGANG MAASS[1]

ABSTRACT. Countably many $\aleph_1$-recursively enumerable sets are constructed from which all the $\aleph_1$-recursively enumerable sets can be generated by using countable union and countable intersection. This implies under $V = L$ that there exists as well a countable basis for Σ_n^1 sets of reals, $n > 2$. Further under $V = L$ the lattice $\mathcal{E}^*(\aleph_1)$ of $\aleph_1$-recursively enumerable sets modulo countable sets has only $\aleph_1$ many automorphisms.

Let $\mathcal{E}$ denote the lattice of recursively enumerable (r.e.) sets under inclusion, and let $\mathcal{E}^*$ denote the quotient lattice of $\mathcal{E}$ modulo the ideal of finite sets. Both structures have been extensively studied (see e.g. the survey by Soare [5]). In recent years research has concentrated on the existence of automorphisms and the decidability of the elementary theory.

Analogous questions arise in α-recursion theory for admissible ordinals α. Here one studies the lattice $\mathcal{E}(\alpha)$ of α-r.e. subsets of α and the quotient lattice $\mathcal{E}^*(\alpha)$ modulo the ideal of α^*-finite sets (see e.g. the survey by Lerman [2]). A set is α-r.e. iff it is definable over L_α by some Σ_1 formula with parameters. A function is α-recursive iff its graph is α-r.e. A set is α^*-finite iff every α-r.e. subset of it is α-recursive. For simplicity we assume $V = L$ in the first part of this paper where we study $\aleph_1$-r.e. sets.

Lachlan has proved the following basic result about automorphisms of $\mathcal{E}^*$ (see Soare [4]): There are $2^{\aleph_0}$ automorphisms of $\mathcal{E}^*$. Sutner [7] has noticed that one can use Lachlan's construction in order to show that for all countable admissible α there are $2^{\aleph_0}$ many automorphisms of $\mathcal{E}^*(\alpha)$. The argument breaks down for $\alpha = \aleph_1$ despite the fact that $\aleph_1$ is like ω a regular cardinal. Observe that in the case $\alpha = \aleph_1$ the α^*-finite sets are just the countable sets. We show in this paper that there are in fact only $\aleph_1$ (instead of $2^{\aleph_1}$) many automorphisms of $\mathcal{E}^*(\aleph_1)$.

DEFINITION 1. We say that a class Γ of sets has a countable basis $(A_n)_{n\in\omega}$ if $\{A_n|n \in \omega\} \subseteq \Gamma$ and Γ is the closure of $\{A_n|n \in \omega\}$ under countable unions and intersections.

Observe that the class of $\aleph_1$-r.e. sets is closed under countable unions and intersections.

Received by the editors June 10, 1980.
1980 *Mathematics Subject Classification.* Primary 03D60; Secondary 03E15, 03D25.
Key words and phrases. α-recursively enumerable sets, automorphisms of r.e. sets, countable unions and intersections of Σ_2^1-sets.
[1]During preparation of this paper the author was supported by the Heisenberg Programm der Deutschen Forschungsgemeinschaft, West Germany.

0002-9939/81/0000-0271/$02.00

THEOREM 2. *The class of $\aleph_1$-r.e. sets has a countable basis $(A_n)_{n\in\omega}$. In fact every $\aleph_1$-r.e. set can be written as a countable intersection of countable unions of countable intersections of the sets $(A_n)_{n\in\omega}$.*

PROOF. Take a universal $\aleph_1$-r.e. set W such that $(W_e)_{\omega\leqslant e<\aleph_1}$ is an enumeration of all $\aleph_1$-r.e. sets, where $W_e := \{\delta | \langle e, \delta\rangle \in W\}$. Further take an $\aleph_1$-recursive function C from $\aleph_1$ into $\mathcal{P}(\omega)$ such that $\{C(e) | e \in \aleph_1\}$ is a family of almost disjoint sets (i.e. every $C(e)$ is infinite and $C(e) \cap C(e')$ is finite for $e \neq e'$, see e.g. Kunen [**1**]).

We construct first countably many $\aleph_1$-r.e. sets $(A_n)_{n\in\omega}$ such that for every $e \in \aleph_1$ with $e \geqslant \omega$

$$W_e - e = \Big(\bigcup_{j\in\omega}\Big(\bigcap \{A_n | n \in C(e) \wedge n \geqslant j\}\Big)\Big) - e.$$

The sets $(A_n)_{n\in\omega}$ are constructed simultaneously in $\aleph_1$ many steps. At step γ we determine for every n on which fact it depends whether or not γ is enumerated in A_n.

We assign in an $\aleph_1$-recursive way to every $\gamma \in \aleph_1$ a function $p_\gamma \in L_{\aleph_1}$ which maps ω one-one onto $\gamma + 1$. For $e \leqslant \gamma$ one might consider $p_\gamma^{-1}(e)$ as the priority of the equality $W_e = \bigcup_{j\in\omega}(\bigcap \{A_n | n \in C(e) \wedge n \geqslant j\})$ at step γ. We change priorities at every step because it is important that the priority list is never longer than ω.

Step γ $(\omega \leqslant \gamma < \aleph_1)$. For $n \in C(p_\gamma(0))$ we determine that γ is put in A_n if and only if γ is enumerated in $W_{p_\gamma(0)}$. For $j > 0$ and $n \in (C(p_\gamma(j)) - \bigcup_{j'<j} C(p_\gamma(j')))$ we determine that γ is put in A_n if and only if γ is enumerated in $W_{p_\gamma(j)}$. For $n \in \omega - \bigcup_{e\leqslant\gamma} C(e)$ it does not matter whether we put γ in A_n or not.

It is obvious from the construction that the sets A_n are $\aleph_1$-r.e. Further for $\omega \leqslant e < \gamma$ we have

$$\begin{aligned}\gamma \in \bigcup_{j\in\omega}\Big(\bigcap \{A_n | n \in C(e) \wedge n \geqslant j\}\Big)& \\ \Leftrightarrow \gamma \in \bigcap \Big\{A_n | n \in C(e) \wedge n & \\ > \max\Big(C(e) \cap \Big(\bigcup \{(C(e')) | p_\gamma(e') < p_\gamma(e)\}\Big)\Big)\Big\}& \\ \Leftrightarrow \gamma \in W_e.&\end{aligned}$$

So far we cannot generate every set W_e with countable unions and intersections from the basis elements without making mistakes at countably many points. Therefore we add countably many further $\aleph_1$-r.e. sets to the constructed basis elements $(A_n)_{n\in\omega}$ which enable us to correct these mistakes. Let f be an $\aleph_1$-recursive function which maps $\aleph_1$ one-one into $\mathcal{P}(\omega)$. Define $\aleph_1$-recursive sets $(R_n)_{n\in\omega}$ by $\gamma \in R_n :\Leftrightarrow n \in f(\gamma)$. We add then the sets $(R_n)_{n\in\omega}$ and $(\aleph_1 - R_n)_{n\in\omega}$ to the basis. For every $\gamma \in \aleph_1$ we have

$$\{\gamma\} = \bigcap_{n\in f(\gamma)} R_n \cap \bigcap_{n\notin f(\gamma)} (\aleph_1 - R_n).$$

Thus we can write every countable set as a countable union of countable intersections and the complement of every countable set as a countable intersection of countable unions of basis elements. Therefore we can correct every mistake on countably many points.

COROLLARY 3. *There are $\aleph_1$ automorphisms of $\mathfrak{E}^*(\aleph_1)$.*

PROOF. It is obvious that one can construct $\aleph_1$ many $\aleph_1$-recursive permutations of $\aleph_1$ which induce different automorphisms of $\mathfrak{E}^*(\aleph_1)$. On the other hand every automorphism Φ of $\mathfrak{E}^*(\aleph_1)$ preserves countable unions and intersections. Therefore Φ is completely determined by the values $(\Phi(A_n^*))_{n\in\omega}$, where $(A_n)_{n\in\omega}$ is a basis for the $\aleph_1$-r.e. sets and $(A_n^*)_{n\in\omega}$ are the corresponding equivalence classes in $\mathfrak{E}^*(\aleph_1)$.

We now leave α-recursion theory and the assumption $V = L$ and turn to descriptive set theory in ZFC. It makes sense to ask whether the classes $\mathbf{\Sigma}_n^1$ and $\mathbf{\Pi}_n^1$ have a countable basis according to Definition 1 since these classes are closed under countable union and intersection. Obviously if $\mathbf{\Sigma}_n^1$ has a countable basis then the complements of the basis elements form a basis for $\mathbf{\Pi}_n^1$ and vice versa. Observe that $\mathbf{\Delta}_1^1$, the class of Borel sets, has a countable basis. If one chooses suitable basis elements one can generate the Borel hierarchy without using complementation.

COROLLARY 4. *Assume $n \geqslant 2$ and ${}^\omega\omega \subseteq L[a]$ for some $a \subseteq \omega$. Then $\mathbf{\Sigma}_n^1$ has a countable basis.*

PROOF. It is well known that for every $m \geqslant 1$ a subset of ${}^\omega\omega$ is $\mathbf{\Sigma}_{m+1}^1$ iff it is $\mathbf{\Sigma}_m$ definable over HC. Under the assumption ${}^\omega\omega \subseteq L[a]$ we have $\mathrm{HC} = \mathrm{HC}^{L[a]} = L_{\aleph_1}[a]$. Thus the $\mathbf{\Sigma}_2^1$ sets are just the sets which are Σ_1 definable over $L_{\aleph_1}[a]$ and for $m \geqslant 2$ the $\mathbf{\Sigma}_{m+1}^1$ sets are just the sets which are Σ_1 definable over $\langle L_{\aleph_1}[a], \varepsilon, P_m\rangle$ with a suitable mastercode P_m. Since one can define a map which maps ${}^\omega\omega$ one-one onto $\aleph_1$ by a Δ_1 definition over $L_{\aleph_1}[a]$, it does not matter whether one considers subsets of ${}^\omega\omega$ or of $\aleph_1$. Further the construction of a countable basis in the proof of Theorem 2 works as well for $L_{\aleph_1}[a]$ and $\langle L_{\aleph_1}[a], \varepsilon, P_m\rangle$ instead of $L_{\aleph_1}$.

REMARK 5. Richard Mansfield has shown [3] that any countably generated σ-algebra consisting entirely of Lebesgue measurable sets does not contain all $\mathbf{\Sigma}_1^1$ sets. Therefore $\mathbf{\Sigma}_n^1$ has no countable basis in the sense of Definition 1 if all $\mathbf{\Sigma}_n^1$ sets are measurable. This implies that $\mathbf{\Sigma}_1^1$ never has a countable basis. Further, Solovay's model of ZFC where all projective sets are measurable [6] supplies an example where no $\mathbf{\Sigma}_n^1$ has a countable basis.

In addition Mansfield has given a complete answer for $\mathbf{\Sigma}_2^1$: If $\mathbf{\Sigma}_2^1$ has a countable basis then ${}^\omega\omega \subseteq L[a]$ for some $a \subseteq \omega$ (to appear).

REFERENCES

1. K. Kunen, *Combinatorics*, Handbook of Mathematical Logic (J. Barwise, Ed.), North-Holland, Amsterdam, 1977.

2. M. Lerman, *Lattices of α-recursively enumerable sets*, Proc. Second Sympos. Generalized Recursion Theory (Oslo, 1977), North-Holland, Amsterdam, 1978.

3. R. Mansfield, *The solution to one of Ulam's problems concerning analytic sets*. II, Proc. Amer. Math. Soc. **26** (1970), 539–540.

4. R. I. Soare, *Automorphisms of the lattice of recursively enumerable sets*. I, *Maximal sets*, Ann. of Math. (2) **100** (1974), 80–120.

5. ______, *Recursively enumerable sets and degrees*, Bull. Amer. Math. Soc. **84** (1978), 1149–1181.

6. R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. (2) **92** (1970), 1–56.

7. K. Sutner, *Automorphisms of α-recursively enumerable sets*, Diplomarbeit an der Universität München, 1979.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139