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RECURSIVELY INVARIANT β-RECURSION THEORY

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Recursively Invariant β-Recursion Theory

(Preliminary Survey)

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In my lecture I want to sketch a new branch of generalized recursion theory: invariant β-recursion theory. β is any limit ordinal in the following.

A set $A \subseteq L_\beta$ is called β-recursively enumerable (β-r.e.) if it is definable over L_β by some Σ_1 formula φ (see Friedman and Sacks [1]). Observe that this is really a very intuitive definition. Generate successively the levels $L_0, L_1, \ldots, L_\gamma, \ldots$ $(\gamma < \beta)$ of the constructible hierarchy up to β. Enumerate at every step γ those elements z into A which satisfy $L_\gamma \models \varphi(z)$ and which have not already been enumerated before.

The example shows that the general concept of a recursively enumerable set -as described by Post [2] in 1944- does not require any strong closure conditions of the underlying domain like admissiblity.

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A function $f : L_\beta \to L_\beta$ is called β-recursive if its graph is β-r.e..

Consider the group of all β-recursive functions which map L_β one-one onto L_β together with composition of maps. A property of subsets of L_β is called G-invariant or recursively invariant if for every $f \in G$ some set $B \subseteq L_\beta$ has this property if and only if $f[B]$ has it.

Felix Klein suggested in his Erlanger Programm (1872) to define branches of mathematics in terms of a space X and a group G of transformations acting on that space. The branch of mathematics determined by X and G is the study of G-invariant properties.

L_β and the previously defined group G determine for $\beta = \omega$ classical recursion theory and for $\beta = \alpha$ (α admissible) α-recursion theory.

Let us now look whether there is an appropriate notion of finiteness in invariant β-recursion theory. Any recursively invariant class of β-recursive bounded (i.e. $\subseteq L_\gamma$ for some $\gamma < \beta$) subsets of L_β is a candidate for such a notion. It is obvious that there exists a largest such class which we call I. We will see in the following that there are several good reasons to take I as the notion of finiteness in invariant β-recursion theory. The elements of I are called i-finite sets. If β is an admissible ordinal α then i-finite is equivalent to α-finite.

Define $\sigma 1 \mathrm{cf} \beta :=$ the least $\delta \leq \beta$ (there exists some β-recursive $f : \delta \to \beta$ with range unbounded in β).

Lemma 1 : I is a β-recursive subset of L_β. In fact
$I = \{x \in L_\beta \mid L_\beta \models [\text{cardinality}(x) < \sigma 1 \text{cf} \beta]\}$.
The proof is not difficult but relies heavily on the fine structure of L (collapsing of Skolem hulls).

Every β-r.e. set A can be enumerated in $\sigma 1 \text{cf} \beta$ many steps, i.e. there exists a β-recursive function $f : \sigma 1 \text{cf} \beta \to L_\beta$, $\gamma \mapsto A_\gamma$, such that $A = \bigcup \{A_\gamma \mid \gamma < \sigma 1 \text{cf} \beta\}$. Thus every single $x \in A$ is enumerated after an i-finite number of steps. It is easy to see that I is the only recursively invariant class of β-recursive bounded subsets of L_β which is in this sense coherent with the notion of a β-r.e. set. Further for any i-finite subset K of A we have $K \subseteq A_\gamma$ for some i-finite γ. This property is important for priority constructions. It implies that every true i-finite neighborhood condition about A settles down at some point of the construction.

Another useful property is the following : Every β-recursive subset of an i-finite set is again i-finite.

Consider for any limit ordinal β the structure $\mathfrak{A}_\beta := \langle L_\beta - I; I, \tilde{\in}, T \rangle$ where $\tilde{\in} := \in \upharpoonright L_\beta \times I$ and T is the canonical β-recursive truth predicate for Δ_0 L_β formulas in L_β. $\mathfrak{A}_\beta$ is construed as a set with urelements as in Barwise [3], where $L_\beta - I$ is the underlying collection of urelements.

Theorem 2 : $\mathfrak{A}_\beta$ is an admissible structure with urelements.
For every set $M \subseteq L_\beta$:

M is $\Sigma_1(\Delta_1)$ L_β <=> M is $\Sigma_1(\Delta_1)$ $\mathfrak{A}_\beta$.

Further the sets in the structure $\mathfrak{A}_\beta$ are exactly the i-finite sets.

Corollary 3 : Assume that β is a countable limit ordinal. Let $\mathcal{L} \subseteq L_\beta$ be some β-recursive language and let T be a β-r.e. set of sentences in the language $\mathcal{L}$ with i-finite disjunctions and conjunctions.
If every i-finite set $T_0 \subseteq T$ has a model, then T has a model.

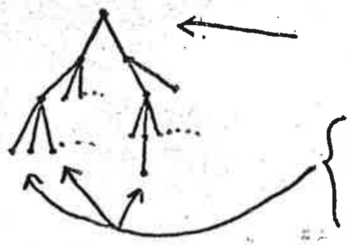
Proof of the Corollary : Apply the Barwise Compactness Theorem [3] to $\mathfrak{A}_\beta$.

Remark : The compactness theorem does not hold for any larger notion of "finite" in L_β .

The preceding compactness theorem (Corollary 3) can be used to show that for every countable β invariant β-recursion theory can be characterized in terms of absoluteness or model theoretic invariance as this effect was called by Kreisel [4]. The concept of model theoretic invariance is useful in order to understand the mathematical meaning of computations in recursion theory. The situation is analogous as in first order logic where the completeness theorem gives a mathematical meaning to formal proofs.

The connection between model theoretic invariance and recursive invariance is the following : The notion of a "finite" set is recursively invariant in every recursion theory which can be characterized in terms of model theoretic invariance.

In order to get an intrinsic notion of a computation relative to an oracle $B \subseteq L_\beta$ one can extend the Kripke equation calculus in a canonical way. The essential rule allows to survey i-finite many subcomputations in a computation. Every computation has the structure of an i-finitely branching tree :



We say that A is computable from B if the characteristic function of A can be computed from B in this equation calculus. We say that A is i-finitely computable from B if this can be done by using i-finite computations only. B is called semigeneric if every equation which can be computed from B can be computed from B with an i-finite computation. For a semigeneric set B the preceding two notions of reducibility coincide for every set A .

<u>Lemma 4 :</u>

a) For countable β A is computable from B iff A is implicitly invariantly definable from B (see [6]).

b) A is i-finitely computable from B iff there exists a β-r.e. set W such that for every $x \in L_\beta$:

$c_A(x) = 1 \leftrightarrow \exists$ i-finite $K,H(\langle x,i,K,H\rangle \in W \wedge K \subseteq B \wedge H \subseteq L_\beta - B)$

(c_A is the characteristic function of A).

c) B is semigeneric iff for every relation $R \subseteq L_\beta \times L_\beta$ of the form

$R(x,y) \leftrightarrow \exists$ i-finite $K,H(\langle x,y,K,H\rangle \in W \wedge K \subseteq B \wedge H \subseteq L_\beta - B)$

with W β-r.e. and dom R i-finite there exists an i-finite function $f \subseteq R$ with dom f = dom R.

The relation in b) is not transitive and therefore we consider instead the following reducibility relation :

$A \leq_i B$:<=> there exists a β-r.e. set W such that for all i-finite H_1, H_2

$H_1 \subseteq A \leftrightarrow \exists$ i-finite $K,H(\langle H_1,1,K,H\rangle \in W \wedge K \subseteq B \wedge H \subseteq L_\beta - B)$

and

$H_2 \subseteq L_\beta - B \leftrightarrow \exists$ i-finite $K,H(\langle H_2,2,K,H\rangle \in W \wedge K \subseteq B \wedge H \subseteq L_\beta - B)$.

The associated equivalence classes are called i-degrees. For admissible α they coincide with the α-degrees.

Every i-degree is recursively invariant. The i-degree 0 (i.e. the equivalence class of the empty set) contains exactly the β-recursive sets. As usual one gets immediately that there exists a maximal β-r.e. i-degree 0' which is strictly greater than 0 . There is no trivial way to show the existence of an intermediate β-r.e. i-degree.

Except for a few β (where it is still open) one can understand the structure of the β-r.e. β-degrees (see[1]) as a substructure of the i-degrees.

<u>Theorem 5 :</u> For every limit ordinal β there exist β-r.e. sets A , B such that $A \not\leq_i B$ and $B \not\leq_i A$.

The <u>proof</u> is given in the most interesting case where β is strongly inadmissible (i.e. $\sigma 1 \mathrm{cf} \beta < \beta^*$) by a priority construction following Friedman [5]. The combinatorial principle $\diamondsuit$ can here be eleminated (this may be helpful for applications to other inadmissible sets).

Observe that for every β the i-degrees coincide with the degrees in the admissible collapse $\mathfrak{A}_\beta$. Thus Theorem 5 contains as a special case the solution of Post's Problem for some enormously fat admissible sets.

<u>Theorem 6 :</u> For every limit ordinal β there exist β-r.e. sets A , B such that A is not computable from B and B is not computable from A .

The <u>proof</u> is slightly more difficult than the proof of Theorem 5 . We make A and B in addition semigeneric. For this one needs $\diamondsuit$.

<u>Theorem 7 :</u> For many strongly inadmissible β there are β-r.e. sets A such that $0 <_i A$ but $S \not\leq_i A$ for every simple set S (see [6] for the definition of simple).

The <u>proof</u> is a first example of an infinite preservation strategy in the strongly inadmissible case. Besides $\diamondsuit$ it uses a new combinatorial argument . We expect that refinements of the applied strategy will lead to a splitting theorem for i-degrees.

We had mentioned the definition of a semigeneric set because at this point an important new effect arises in the step from α- to β-recursion theory. Several equivalent definitions of "hyperregular" in α-recursion theory lead to different classes in β-recursion theory. For some strongly inadmissible β there are β-r.e. sets B such that every computation from B has an i-finite length but B is not semigeneric.

All details can be found in the forthcoming paper [6].

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